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An Experiment in Linear Space Perception

BY

FRANCIS NORTON MAXFIELD

A Thesis presented to the Faculty of the Graduate School of the University of Pennsylvania in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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EXPERIMENTAL PROCEDURE

The purpose of the experiment under consideration has been to apply the statistical methods outlined in F. M. Urban's monograph on Statistical Methods in Psychophysics[1] in another field, viz., the discrimination of small differences in line lengths. Urban's methods have thrown new light on psychophysical problems and especially upon the interrelation of the different methods of psychophysical experimentation and the interpretation of results. His work is based upon experiments with lifted weights. As the present experiment was intentionally arranged to lend itself to similar treatment, much of the detail of laboratory procedure was made to conform, as nearly as possible in a different field, to the plan of the earlier experiments.

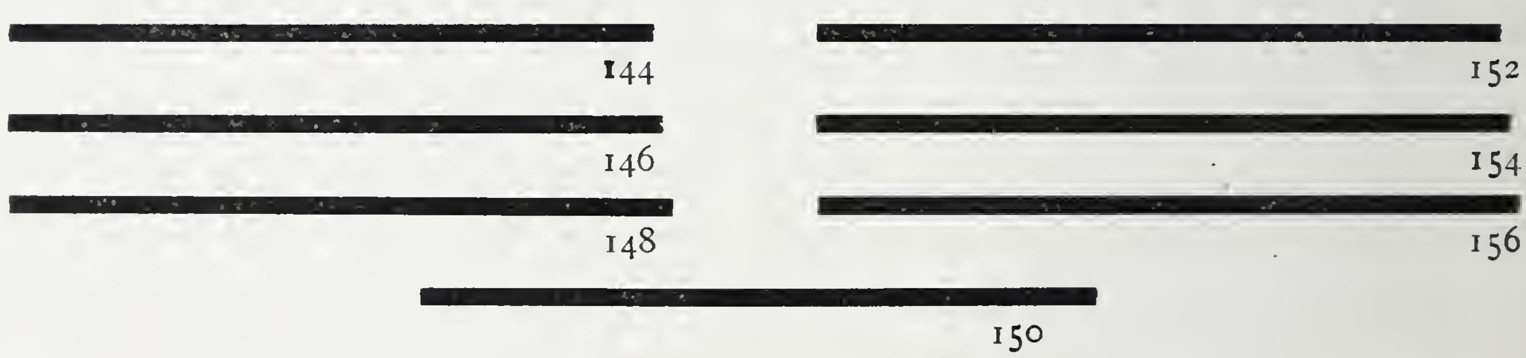
Urban used seven comparison weights of 84, 88, 92, 96, 100, 104, and 108 grams respectively. These were placed along the edge of a circular turning-table 3 feet in diameter alternately with seven standard weights of 100 grams each. The subject sat at the table so that his hand was just above each weight in turn, as the table was revolved by the experimenter. He lifted a standard weight and then a comparison weight, making the judgment "heavier", "lighter", or "equal", which was recorded by a third person. The details of this procedure are given fully in the monograph referred to above and where there has been an important departure from them in the present procedure, the reasons for the variation are indicated below.

The judgments given in this experiment were based on a visual comparison of black lines printed on white cards and presented to the subject in pairs. Seven comparison lines were used. Each pair, a comparison line and a standard line, was printed on a separate card 3½ by 12 inches, lengthwise of the card, in its median line, and one inch apart. Care was taken to center

[1] The Application of Statistical Methods to the Problems of Psychophysics; F. M. Urban, Ph.D. (Psychological Clinic Press, Philadelphia, Penna. 1908.)

these so that the adjacent ends of the lines were each ½ inch from the center of the card. The variation of the distance between the end of the line and the end of the card was less important, amounting to only about 2 per cent. Moreover the space between the lines was in the field of attention, whereas the other spaces were not.

A standard stimulus of a black 4-point line, 150 printers' "points" long, was used. There were seven comparison stimuli, also 4-point lines, of 144, 146, 148, 150, 152, 154, and 156 points respectively. These lines were printed on white cardboard, several sets being prepared so that it might not be necessary to use soiled or marked cards. These lengths were chosen after extensive preliminary experiments with more extended series having greater differences between the comparison stimuli. The printers' point (1/72 inch) was used as a unit of measurement because the lines could then be cut more accurately by the cutting-machine of a printers' supply-house than by the hand methods necessary if any other scale had been used. The difference between the two comparison lines at the extremes of the series is therefore ⅙ of an inch, or 8 per cent of the standard in comparison with Urban's range of 24 per cent in lifted weights. A 4-point line was chosen on the ground that a wider line would introduce an area judgment, the line appearing as a black rectangle, and that slight variations of illumination or of optical defect would become disturbing factors with a narrower line. Specimens of the lines used are given here.



These cards were presented to the subject by a machine designed particularly for this purpose. A drum of light frame in the form of a fourteen-sided prism was rotated by an inter-

mittent gear connected with a speed-reducer and motor. A card was fastened in place upon every other face of this prism, the intervening spaces being covered with white paper to match the cards. Each card in succession was brought to rest for about 1½ seconds behind an opening 2½ by 10 inches in a cardboard screen which cut off all view of the apparatus from the subject who sat at a small table seven feet from the screen. The opening was on a level with the subject's eyes. The screen and card were lighted by a large window behind the subject. There were no other windows in the room, the walls were of a neutral tone, and there was little to distract the attention of the subject. There was a slight whirring noise from the motor and a click as the drive-wheel picked up the gear-wheel on each revolution; otherwise the room was quiet.

The cards were held firmly in place on the drum but could be withdrawn easily when it was necessary to replace them or to change the order. The seven card positions were marked plainly with the numbers 1 to 7 on the end of the drum. Five series of cards were prepared, each being made up of seven similar cards arranged in a different order. Each card in a given series was inconspicuously marked at one end with its series number and with the number of the face of the drum on which it was placed (e. g., Series III, Number 6), but bore no mark by which a subject could identify it when examining the apparatus. These marks were at the extreme margin and so could not be seen by the subject during the experiment. As the marked end of the card was always placed toward the numbered end of the drum, it was easy to avoid placing a card in a wrong position.

As the subject faced the apparatus he was directed to lean forward, sit up straight, or lean back in his chair according as one or the other position seemed to give him the best view of the lines. If he ordinarily wore glasses, he retained them. He was told to direct his attention to the *left-hand* line, the standard, and then to carry his eyes across to the right and give his attention to the other line, the comparison stimulus. He then pronounced aloud his judgment as "longer", "shorter", or "equal",

the record being made by a second person. All subjects used the word "equal" but some preferred the word "greater" for "longer" judgments and "less" for "shorter" judgments. Each was encouraged to use the word which most promptly and spontaneously fitted his judgment. In a few cases a subject would revise his judgment, saying for example: "Longer; no, I mean equal." In such cases the second or final judgment was recorded. In the early conduct of the experiment a few of the subjects showed a tendency to omit judgments on single pairs of lines here and there in a series. It was necessary to caution recorders to observe closely that all seven were judged. Where one of a series of seven was omitted, the remaining six judgments on the pairs 1 to 7 were thrown out. The recorder sat where he could control the motor-switch and watch the apparatus; and particularly so that he could see the numbers on the drum. The subject began to give judgments as soon after he was seated as it was possible for him to get a good view of the opening in the screen or, if the apparatus was not in motion, as soon as the movement began. Wherever in a series a subject began to give his judgments, the recorder always made his first record on the pair numbered 1. In this way it was possible for subject and recorder to interchange places since neither knew the order of the comparison stimuli and since the observer could not even tell when he began to give judgments on the first comparison pairs. On the record sheets the pairs were known only by their series and order numbers so that two men acting alternately as subject and recorder could have completed all five series without knowing either the absolute or the relative lengths of the comparison lines. However the experimenter himself recorded some of the judgments of most of the subjects and in some cases all of them. He took special pains to do this in cases where there appeared to be something unusual in the general trend of the subject's judgments. The records were made on sheets printed for this purpose. Each record-sheet had a space at the top for the number of the series, the name of the subject, and the date of the experiment. Seven lines, numbered from 1 to 7, were carried across the page and

were divided into squares so that the recorder filled in the squares of one column and then of the next as the cards numbered 1 to 7 came into place and were judged by the subject. The letters *g*, *e*, and *s* were used for longer, equal, and shorter, respectively, *g* being used in place of *l* since preliminary experiments showed that *l* was easily confused with *e* in the handwriting of some recorders. When ten, or in some cases fifteen, of these columns had been filled, the subject was allowed to rest or exchanged places with the recorder. The following is a transcript from an actual record-sheet.

Name—Mr. Smith *Series*—III *Date*—Mar. 25, 1911

1	s	g	g	e	g		g	g	g	g	g		g	s	e	g	e			
2	g	g	s	e	e		g	s	s	e	g		e	s	s	s	g			
3	g	g	g	g	g		g	g	g	g	s		g	g	g	g	g			
4	g	g	g	s	s		g	s	g	g	g		g	g	g	s	g			
5	g	g	g	g	g		g	g	g	g	g		g	g	g	g	e			
6	g	g	g	g	g		e	g	s	g	s		e	g	g	g	g			
7	s	s	s	s	s		s	s	s	s	s		s	s	s	g	s			

In the preliminary experiments various devices were tried by which the subject might record his own judgments, but none of them proved in the least degree satisfactory. On the whole the method employed commends itself for use in future experimentation.

Eleven of the twelve subjects were men. The other was a woman, a teacher in a private-school near Philadelphia. Of the men, one was Dr. Urban himself, to whose sympathetic co-operation the writer is indebted, two were graduate students, and eight undergraduate students at the University of Pennsylvania. Their ages probably ranged from eighteen to thirty-five years. Six of the undergraduate students were employed as subjects and were paid for their services. The other six were in a sense voluntary subjects although three of them took part in this experiment as a part of their psychological laboratory work in course.

Where subjects had no previous knowledge of the nature of the experiment, little attempt was made to explain its different phases. Enough information was given to elicit as high a degree of attention as possible and very careful and particular directions were given in regard to the manner of looking at the cards and of giving judgments. Though the experiment was not concerned with the *rightness* or *wrongness* of the judgments, subjects were sometimes told that they were "doing well' or that they were "improving" when this was the case, or that "one man was doing better than another". The purpose of this was to secure the highest degree of attention. The subjects were not informed that the seven cards of one series were the same as the seven cards of another but in different orders, though many of them knew it. Nor on the other hand was such information withheld where subjects became interested in the experiment. In some cases subjects were shown the results of their first series before later series were completed. Although some of the subjects were employed, the experimenter has every reason to believe that, as far as attention could be controlled, each subject did his best to concentrate upon the experiment and to follow out the instructions in regard to making judgments. For reasons discussed later, no effort was made to train subjects for this experiment though three of the subjects had had a considerable amount of training in psychological laboratory work. As will appear later in the discussion of results, there are some drawbacks to having subjects with no training on the particular experiment undertaken even if they may have had training in other psychological laboratory work. Training was omitted in this series of experiments partly on account of the difficulty of keeping the training element the same in all subjects and partly because in these experiments it seemed desirable to see to what extent they could be used as a sensitivity test of a new subject with no training.

The seven cards, as previously stated, were arranged in five series. The orders of the comparison lines in these series were as follows:

	Series I	Series II	Series III	Series IV	Series V
1.	146	156	152	150	144
2.	154	148	146	156	152
3.	150	146	154	152	156
4.	156	152	148	146	146
5.	144	144	156	144	150
6.	148	154	150	148	154
7.	152	150	144	154	148

Since a subject did not know when he was judging Number 1, Series I might just as well read 150, 156, 144, 148, 152, 146, 154, 150, 156, 144, etc. All the subjects testified that they had no sense of a series of seven beginning at a particular card and again beginning there. This was true even with those subjects familiar with the general plan of these series.

Each subject made 100 judgments on each card of each series, of 3500 judgments in all, so that the total number of judgments for all subjects was 42,000. These experiments were made during different periods varying for different subjects from five to seven weeks. As explained above, subjects were instructed to sit facing the opening in the screen and to assume the best position for distinct vision. They were to look first at the left-hand line or standard stimulus and then at the right-hand line or comparison stimulus and then to give the judgment "longer", "equal", or "shorter". No attempt was made to get any record of the subject's feeling of certainty in regard to these judgments nor were any "guesses" recorded in the case of the pairs judged equal. As the feeling of certainty is bound to be more or less a relative thing when the subject comes to express it in words, and as to require an introspective judgment on a judgment tends to distract attention from the primary judgment which is the object of the experiment, it did not seem wise to pay attention to the matter of certainty. The subjects were told to try hard to see whether the right-hand line was longer or shorter, but failing to see that it was either, to give the judgment "equal". This idea of subjective equality was explained and insisted on. It is true that some observers consider that there is a distinction between inability to see which is longer of two lines (doubtful case) and the judgment "equal" which is a category correspond-

ing to "longer" or "shorter" when given with some feeling of certainty. It seemed best to make no distinction between these two classes since it seems to the experimenter that the "doubtful cases" and the equality judgments are both equality cases in that the observer fails to perceive the difference between two stimuli which are presented to him. When observers speak of a difference of which they cannot determine the direction, as when two lines do not appear to be the same length and yet the observer is unable to tell which seems longer, it seems probable that there is some fluctuation in the "sensation of length" due to variations in objective or subjective conditions. For example, the influence of immediate fixation may be felt. A subject might look at the standard line and then at the comparison line. The latter might seem slightly longer but so uncertainly that the subject's eyes would return to the standard which when fixated might itself seem the longer of the two; that is, each line in its turn seems longer than the other. This is a true "doubtful case". From the standpoint of this experiment, however, it seemed best to insist on the definition of subjective equality given above.

While all subjects apparently tried to use the equality judgment as directed, there was much variation in its actual use. It would seem that this variation was greater than would be accounted for by subjective differences in different subjects and yet an effort was made to keep outside conditions in the way of directions, explanations, etc., the same for all. Seventeen per cent of all judgments given were of the equality type as defined above. These varied from less than 4 per cent in the case of one subject to 29 per cent in another.[2] There was also a great variation in the distribution of these judgments among the different pairs and in the different series. Subject X gave 264 equality judgments but 155 of these were in the first series and Subject VI gave 1022, with 536 in his first two series. These variations suggest that a series might be given in which a subject

[2] Urban's subjects gave 14 per cent. of all judgments as "equal" with a range of 5 per cent. to 33 per cent. His subjects show less variation from series to series. This is certainly one of the advantages of training one's subjects.

might make the judgment "longer" or "not longer" on 100 pairs of comparison stimuli, then the judgment "equal" or "not equal" on the next 100 pairs, and on the third hundred the judgment "shorter" or "not shorter". The difficulty of giving uniform directions to inexperienced subjects would be lessened and the choice between two judgments is of course simpler than among three for either trained or untrained subjects.

Compare this simplicity with the complications introduced by requiring "guesses" and introspections as to "degrees of certainty" on primary judgments. Take for example the series sometimes used:

"smaller"	(certain)
"smaller"	(less certain)
"doubtful"	(probably smaller = "guess")
"equal"	(certain)
"doubtful"	(probably greater = "guess")
"greater"	(less certain)
"greater"	(certain)[3]

Instead we might almost give each of these seven judgments a number corresponding to the seven values of our comparison stimuli taken in order and then ask the subject to call or try to call each comparison line as it is presented by its appropriate number. If it was "certainly shorter" he would say "144", if "less certainly", "146", if "doubtful but probably shorter", he would say "148", etc. Or a subject might adopt the visual scheme mentioned by Martin and Müeller.

In this scale zero stands at the middle and corresponds to subjective equality. To the left are the smaller judgments and to the right the greater. The distance from the center is a measure of the certainty of a judgment in any particular part of this scale.

[3] Compare Urban's h_1, h_2, etc.

Thus *kl* is "certainly smaller", kl is "smaller", glkl is "equal, probably smaller", etc. The subject gives his judgment a place in this visual scale and then gives the appropriate answer.[4] But either this plan or the previous one involves complications in judgment where simplicity is desirable. Urban well says (*op. cit.* p. 15) "that there is some danger of mixing up two different problems, the problem of the degree of subjective confidence and that of the accuracy of sensation".

Subjects were encouraged to give judgments promptly but not hurriedly. They were told to avoid as far as possible the inevitable tendency to compare the comparison line on which judgment was being made with the previous comparison line which had occupied the same position in the field of vision rather than with the accompanyng standard line.[5] Although the subjects were told to view the standard line first and the comparison line afterward, there was also a tendency to let the eyes pass back and forth from one to the other. In the long run the influence of the previous comparison line in the judgment of any pair of lines is probably appreciable, but not serious in the case of subjects who are conscientiously trying to carry out the plan of the experiment if care is taken to change the serial order often enough so that this influence may not always be in the same direction. As brought out later, it is probable in this experiment that it would have been better to have used ten rather than five serial orders. The following figures show that in cases where the objective difference of a pair of comparison lines is small this influence is likely to be effective. In series III comparison line 150 follows comparison line 156, while in series V 150 follows

[4] Zur Analyse der Unterschiedsempfindlichkeit, L. J. Martin und G. E. Müller, Leipzig, 1899; p. 50. Dr. Henri, the subject who used this scale, said "Ganz unwillkurlich hat sich bei mir ein visuelles Schema für die Antworten ausgebildet. Das Schema besteht aus einer Geraden. In der Mitte ist Null, rechts die Antworten 'grösser', links die Antworten 'kleiner'. Manchmal, nachdem Ich das zweite Gewicht gehoben habe, stelle Ich nur visuell die Antwort vor und nur etwas später kommt die Antwort ins Bewusstsein. Dieses scheint meistens in den Fällen glkl und glgr zu geschehen."

[5] Nebenvergleichungen.

146. Taking all twelve subjects into account, the results are as follows:

Comparison line 150 compared with standard line 150 was judged

	shorter or longer;	shorter;	longer;
In Series III	980	564	416
In Series V	984	488	496

It will be observed that the number of equality judgments (220 in series III and 216 in series V) remains practically the same while the change in the other two classes of judgments is apparent. This comparison suggests that the previous appearance of line 156 makes a subject more likely to underestimate line 150, while the previous appearance of line 144 or 146 makes a subject more likely to overestimate this same line.

Several subjects spoke of the influence in some cases of an absolute impression. A certain line, either standard or comparison, would seem "short" or "long" by itself alone and that without conscious reference to the accompanying or preceding lines. Again a pair of lines would seem "short" and in the case of one subject a whole series seemed "shorter than any previous series". These absolute impressions are a factor in the formation of our judgments and in some cases must exert an appreciable influence.

One revolution of the drum required about 18 seconds. Each card was presented for about 1½ seconds and the change to the next required 1 second. This change was rapid enough to make one pair of lines seem to disappear at once; then there was a blank; and then a second pair appeared. There was little to suggest a moving stimulus though, of course, the lines did move in the field of vision 1 inch before and 1 inch after they were at rest. The simplicity of the apparatus commends itself in that neither experimenter nor subject need give attention to its manipulation. A detailed description follows on page 41.

Every effort was made to avoid illusions of any kind and, where time and space errors could not be avoided, to keep these as constant as possible. To this end the lines were placed in the horizontal plane and moved downward to be replaced by the next pair. Each line was at the same distance from the observer as the

others. No effort was made to have the subjects remember the standard from one card to another. Subjects were not even asked to remember that, objectively speaking, the standard lines were all the same. In other words, each pair was to be judged for itself as it came along.

At least two other procedures might obviously have been pursued. The standard line might have been put at random, now at the right, now at the left. The subject would then have been asked to pick out the longer of each pair. Again the standard line might have been placed at the right throughout the experiment, reversing our actual procedure. The former change would have made it more difficult to summarize the records accurately and it would have been less feasible to have subjects act in turn as recorders. The left-to-right method was preferred to the reverse in that the left-to-right excursion of the eyes is the customary one in reading.

No attempt was made to emphasize *rightness* or *wrongness* of judgments except as noted above in the matter of securing attention. Occasionally during times of rest, a subject would speak of a series as "difficult" or "easier than the previous series" but the correctness of his answers when finally counted seldom bore him out in his idea that he was doing better. If a subject knows that he is dealing with seven pairs only one of which has the quality of objective equality, he is very likely to be influenced by his knowledge the moment emphasis is put on the correctness of his judgments. Nor is this matter of correctness of particular interest to the experimenter except as it gives evidence of some constant "error" of observation. The experimenter is much more concerned with his subject's ability to report accurately and promptly subjective equality and subjective difference. This again suggests the advantage of having trained subjects. On the other hand, if psychophysical methods are ever to find extensive clinical or laboratory use, it will be necessary to devise methods of testing untrained subjects and either to record many experiments in a short time or be satisfied with few.

ABSOLUTE FREQUENCY

The record-sheets of each subject were collected and the number of each kind of judgments made upon each comparison line in each half series (50) was found and marked on the record-sheets. The results of these countings are the absolute frequencies of the different classes of judgments for each comparison stimulus and are tabulated on pages 50 to 53. There are considerable variations in the different series and some even in the two fifties (a and b) of a single series. Since other conditions were kept as nearly constant as possible, these variations are in many cases probably due to some influence of the order in which the pairs were presented in the different series. As stated above, there seems to be a tendency to compare a comparison line with the preceding comparison line rather than with the accompanying standard line. It is therefore of interest, in comparing the different series of a single subject or in comparing the judgments of different subjects one with another, to form some estimate of the extent to which the conditions under which these data have been derived have been kept constant. If we make s observations in each of n series of a certain event E of which m_1, m_2, m_3, - - - - - - m_n have given a certain result, the relative frequencies of these results will be represented by the quotients

$$\frac{m_1}{s}, \frac{m_2}{s}, \frac{m_3}{s}, \quad . \quad . \quad . \quad \frac{m_n}{s}.$$

It the sum of these ratios be divided by n we derive a, the arithmetical mean; that is

$$a = \frac{\Sigma m_i}{ns}$$

If a large number of observations are made, this arithmetical mean may be taken as the mathematical probability of the event E. As for example, if successive drawings are made from the same urn containing a certain number of black and white balls, and the balls are returned after each drawing, the relative frequencies of a given result will be grouped around a certain value p with a

coefficient of precision h which may be obtained from the equation

$$h=\sqrt{\frac{s}{2p(1-p)}}$$

according to the theorem of Bernouilli. In the same way we may expect, under constant conditions in the collection of experimental data, a coefficient of precision

$$h=\sqrt{\frac{s}{2a(1-a)}}$$

If we treat our results m_1, m_2, m_3 ------ m_n by the method of least squares, we may compute a coefficient of precision h' according to the following formula:

$$h'=\sqrt{\frac{n-1}{2\Sigma\left(\frac{m_i}{s}-a\right)^2}}$$

A comparison of the two values, h and h', gives an idea of the extent to which the conditions of the experiment have remained constant, since in that case the ratio $\frac{h}{h'}$ approximates 1. This ratio is called the *coefficient of divergence* and we have

$$Q=\sqrt{\frac{\Sigma\left(\frac{m_i}{s}-a\right)^2 s}{a(1-a)(n-1)}}$$

In the present experiment, this calculation was made for all twelve subjects with reference to the distribution of the relative frequencies for shorter, equal, and longer judgments. The results appear on page 55. Since $n = 10$ and $s = 50$ for all twelve subjects, we have the following convenient arrangement for this formula, for the values $\frac{m_i}{100}$, etc., and $\frac{a}{2}$ may be obtained from the tables of absolute frequencies by a mere adjustment of decimal points.

$$Q=\sqrt{\frac{200\Sigma\left(\frac{m_1}{100}-\frac{a}{2}\right)^2}{9a(1-a)}}$$

It is of interest to study the values thus obtained . If we average the coefficients of divergence for all twelve subjects, we obtain the following results:

c.s.	shorter	*	equal	*	longer	*
144	2.09	+0.25	1.96	—0.25	1.54	—0.47
146	2.05	+0.21	2.05	—0.16	1.84	—0.17
148	2.33	+0.49	2.50	+0.29	1.94	—0.07
150	1.82	—0.02	2.24	+0.03	2.26	+0.25
152	1.64	—0.20	2.42	+0.21	2.37	+0.36
154	1.43	—0.41	2.14	—0.07	1.99	—0.02
156	1.50	—0.34	2.14	—0.07	2.15	+0.14
Mean	1.84		2.21		2.01	

We may observe that, from the standpoint of a "normal" distribution, these coefficients are all large. This is not surprising when we consider that none of the subjects was trained in this particular experiment, and that six of the twelve had had no training in similar experimentation. As we might expect from our observations above on the distribution of the equality judgments, the coefficients for these judgments give a mean (2.21) clearly larger than that for either shorter or longer judgments (1.84 and 2.01 respectively). It will be observed that the coefficients for shorter judgments are largest for the shorter comparison stimuli, 144, 146, and 148; those for equality judgments are largest for comparison stimuli in the middle of our series, 148, 150, and 152; while those for longer judgments are largest for the latter part of the series, 150, 152, 154, and 156.

In this connection it is of interest to compare the coefficients of different subjects. It will be remembered that the six subjects who were employed were not only less experienced but also younger and less mature than the other six. If we arrange the six inexperienced subjects in one table and the other six in another we have the following result:

(* Differences from mean for all seven comparison stimuli.)

(A) Six inexperienced subjects

	s	e	l
I	2.14	3.27	2.09
III	2.75	1.99	2.53
IV	2.19	2.93	2.60
V	1.75	2.40	2.27
VIII	2.21	1.94	1.82
X	1.68	3.27	1.97
mean	2.12	2.63	2.21
		2.32	

(B) Six experienced subjects

	s	e	l
II	1.45	2.37	2.46
VI	2.19	2.02	2.55
VII	1.38	1.41	1.25
IX	1.59	1.73	1.31
XI	1.37	1.31	1.38
XII	1.37	1.77	1.77
mean	1.56	1.77	1.79
		1.70	

It may be noticed at once that the several values and averages of the coefficients of divergence for Group A are larger than those for Group B and that the greatest differences and greatest lack of uniformity are in the equality column. The inexperience of the members of Group B may in all probability account for the higher values and lack of uniformity shown. To study this further it has been interesting to study Subject I of Group A. Coefficients of divergence were calculated for the last 2,800 of his 3,500 judgments (*i.e.* Series II, III, IV, and V) and compared with those of the entire 3,500. We find the figures to be as follows:

	For entire 3,500		For last 2,800	
		shorter		
	2.81	144	1.56	
	3.19	146	1.34	
	2.25	148	2.07	
mean 2.14	2.28	150	2.01	mean 1.65
	1.82	152	1.79	
	1.27	154	1.32	
	1.37	156	1.49	
		equal		
	3.28	144	1.81	
	3.77	146	1.44	
	3.16	148	2.06	
mean 3.27	3.32	150	2.46	mean 2.09
	3.83	152	2.98	
	3.21	154	2.53	
	2.32	156	1.32	
		longer		
	1.15	144	1.04	
	1.72	146	1.62	
	2.23	148	2.10	
mean 2.09	1.69	150	1.05	mean 1.76
	3.15	152	2.71	
	2.35	154	2.08	
	1.48	156	1.74	
	mean 2.50		1.83	

These figures plainly show that the conditions under which the last 400 judgments on each comparison pair were made were more nearly constant than for the entire experiment. Subjects III, V, and X from Group A show the same tendency to "steady down" after the first series.

If we arrange our twelve subjects according to the mean value of the coefficients of divergence for all three classes of judgments we have the following result:

1)	VII	1.34	7)	V	2.14
2)	XI	1.35	8)	VI	2.25
3)	IX	1.54	9)	X	2.31
4)	XII	1.62	10)	III	2.42
5)	VIII	1.99	11)	I	2.50
6)	II	2.09	12)	IV	2.57

Out of the first six, five are from Group B. On the basis of his last four series Subject I moves up from eleventh to fifth place. This bears out our previous conclusion that experience is an important factor in experimentation of this kind. This should not imply the element of practice in the usual sense of the word, *i.e.* that, after making many judgments, Subject I could judge more "accurately" of the objective relations of the comparison lines. Such use of the word would imply that, making a mistake, the subject was told of it and so was less likely to repeat his error. It is rather practice in reporting promptly and accurately the subject's own (subjective) judgment. The subjects of Group B on the whole did this better from the first than those of Group A.

This point is illustrated in the case of subject VII who judged comparison line 156 longer than the standard (150) only 340 out of 500 times but with a coefficient of divergence of 1.00. This was not accurate from an objective standpoint but represents good ability in conforming to the conditions of the experiment in a uniform manner and a prompt and accurate introspection as to the subjective state ensuing upon an observation.

The high values of the coefficients of divergence for all twelve subjects for all three classes of judgments suggest that the change of condition due to the order of the comparison pairs in the different series has been effective and that it would have been

better to have used more arrangements, perhaps ten instead of five. The causes of variation for one pair could then be made more nearly the same as for any other. As it is in the doubtful cases that the serial order would be most likely to produce an effect, it is not strange that the averages of the coefficients of divergence of all subjects on the different comparison pairs are largest for comparison lines 152, 150, and 148 as shown in the following table.

c.s.	
144	1.86
146	1.96
148	2.26
150	2.11
152	2.14
154	1.85
156	1.93

On the other hand, if we had taken pains to reduce these large coefficients by practice, or if the serial order had been subject to frequent change, we should have failed, very likely, to have had any evidence other than *a priori* on the advantage of training.

RELATIVE FREQUENCY

Having tabulated the absolute frequencies of the different kinds of judgments for the different subjects, we may proceed to determine the relative frequencies of these judgments. Since 500 judgments were made on each comparison line, these frequencies may be found in each case by multiplying the absolute frequencies by 0.002. The results appear on pages 54 and 55 opposite the numbers 144, 146, etc. These may be considered as empirical determinations of the values of the psychometric functions. Intermediate values of these functions may be interpolated by Lagrange's formula.

This interpolation is based on the assumption that the relative frequency or probability of any judgment is an algebraic function of the objective relations of the comparison stimuli used in an experiment of this nature. If a graded series of compari-

son stimuli x_1, x_2, x_3, ------ x_n give relative frequencies p_1, p_2, p_3 ------ p_n, we may assume that $p = f(x)$ between the limits within which $x_1 > x < x_n$. Then by Lagrange's formula we have

$$p = \frac{(x-x_2)\ (x-x_3)\ \ldots\ldots\ldots\ (x-x_n)}{(x_1-x_2)\ (x_1-x_3)\ \ldots\ldots\ (x_1-x_n)}\ p_1$$

$$+ \frac{(x-x_1)\ (x-x_3)\ \ldots\ldots\ldots\ (x-x_n)}{(x_2-x_1)\ (x_2-x_3)\ \ldots\ldots\ (x_2-x_n)}\ p_2 + \ldots.$$

$$\ldots + \frac{(x-x_1)\ (x-x_2)\ \ldots\ldots\ldots\ (x-x_{n-1})}{(x_n-x_1)\ (x_n-x_2)\ \ldots\ldots\ (x_n-x_{n-1})}\ p_n$$

In the present experiment, this formula gives us the psychometric function p for any comparison stimulus x between the limits within which $144 < x < 156$. Then on the basis of our empirical data, p_1, p_2, etc., we have

$$p = \frac{(x-146)\ (x-148)\ (x-150)\ (x-152)\ (x-154)\ (x-156)}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12}\ p_1$$

$$- \frac{(x-144)\ (x-148)\ (x-150)\ (x-152)\ (x-154)\ (x-156)}{2 \cdot 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}\ p_2$$

$$+ \frac{(x-144)\ (x-146)\ (x-150)\ (x-152)\ (x-154)\ (x-156)}{4 \cdot 2 \cdot 2 \cdot 4 \cdot 6 \cdot 8}\ p_3$$

$$- \frac{(x-144)\ (x-146)\ (x-148)\ (x-152)\ (x-154)\ (x-156)}{6 \cdot 4 \cdot 2 \cdot 2 \cdot 4 \cdot 6}\ p_4$$

$$+ \frac{(x-144)\ (x-146)\ (x-148)\ (x-150)\ (x-154)\ (x-156)}{8 \cdot 6 \cdot 4 \cdot 2 \cdot 2 \cdot 4} p_5$$

$$- \frac{(x-144)\ (x-146)\ (x-148)\ (x-150)\ (x-152)\ (x-156)}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2 \cdot 2} p_6$$

$$+ \frac{(x-144)\ (x-146)\ (x-148)\ (x-150)\ (x-152)\ (x-154)}{12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} p_7$$

Putting c_1, c_2, c_3 - - - - - - c_7 for these fractional coefficients, we have

$$p = c_1p_1 + c_2p_2 + \text{- - - - - -} + c_7p_7.$$

Since these coefficients are the same for all subjects, we find that the logarithmic calculation for interpolation for the values $x = 145$, $x = 147$, etc., is facilitated by first calculating log c_1, log c_2, - - - - - - and log c_7 for each value of x for which it is desired to interpolate. These may then be combined with log p_1, log p_2, etc., which are different for different subjects. The fact that the sum of the values of the three psychometric functions corresponding to the judgments "longer", "equal", and "shorter" for any comparison stimulus x is equal to 1 serves as a convenient check for the values obtained.

The results of these interpolations appear in the same tables with the observed relative frequencies on pages 54 to 55 opposite the numbers 145, 147, 149, 151, 153, and 155. In most cases the interpolated values are intermediate between adjacent empirical values of the psychometric function. In some cases this is not true and we have fluctuating values corresponding to successive comparison stimuli. These are usually near one or the other extremes of our series of comparison pairs and are most marked where $x = 145$ and $x = 155$. In five of the subjects (I, IV, VII, VIII, and XII) one of the interpolated values of the psycho-

metric function for the comparison stimuli 145 or 155 is either greater than unity or less than zero. These values can not, of course, be considered as probabilities. They show rather that the conditions at the extremes of our series are not the same as in its middle course. This corresponds with the evidence of introspection and with our observations upon the coefficients of divergence.

This difference between the conditions of judgment on the extremes of a series of comparison stimuli and on those stimuli more nearly equal to the standard has been pointed out by Urban in connection with his experiment with lifted weights.[1] He formed the mean of the coefficients of divergence for each of his seven subjects on the different comparison weights and for the three classes of judgments and noted that in all three classes the values of the coefficients of divergence were larger for the middle comparison weights of his series. If we do the same for the five subjects mentioned above, we arrive at the following result:

Mean values of coefficients of divergence for subjects I, IV, VII, VIII, and XII.

c. s.	shorter	equal	longer
144	1.76	1.94	1.25
146	2.18	2.43	1.43
148	2.48	2.81	1.65
150	2.12	2.36	2.45
152	1.79	2.36	2.62
154	1.34	2.01	1.92
156	1.31	1.95	1.87

These also give evidence that, where the difference between comparison stimulus and standard is small, the conditions of judgment are not the same as at the extremes of a series where the differences are greater.

A study of the psychometric functions themselves is of interest. That for shorter judgments has a value of about 0.85 for comparison line 144 and, for most subjects, becomes gradually smaller

[1] Die Psychophysischen Massmethoden als Grundlagen empirischen Messungen, F. M. Urban, Ph.D., 1909; p. 280.

throughout the series, being less than 0.10 for comparison line 156. The psychometric function for longer judgments begins at some value less than 0.10 for most subjects and increases throughout the series, gaining for comparison line 156 a mean value of 0.82. These two series of values are more or less symmetrical, one increasing as the other decreases. The psychometric functions for equality judgments vary greatly among the different subjects and will be discussed more at length by themselves. In general they have values nearly equal to zero for comparison lines at the extremes of our series and attain their maximum values near the middle of the series.

A glance at these tables of the numbers of relative frequency will show that, for each subject, for values of x up to a certain point, there is a probability greater than 0.5 that the judgment "shorter" will be given and that, beyond a certain value of x, there is a probability greater than 0.5 that the judgment "longer" will be given. For values of x within this interval there is no probability greater than 0.5 that a comparison stimulus will be judged either shorter or longer than the standard. This is the *interval of uncertainty*. It corresponds to the "difference threshold" of the gradation methods.

Since the values of the psychometric functions increase or decrease with considerable regularity near the middle of our series, we may calculate the limits of this interval of uncertainty by interpolation as of linear functions. For this purpose we find in the tables the values of x corresponding to the values of p which are nearest above and nearest below 0.5 respectively and compute an intermediate value of x for which $p = 0.5$ for shorter judgments. The process is then repeated for longer judgments and the difference between the two results gives the interval of uncertainty in the same units as we measure the comparison stimuli.

Thus if x_a gives probability p_a, and x_b gives probability p_b,

where $p_a > 0.5 > p_b$,

$$\text{let } m = \frac{(x_b - x_a)\ \ (p_a - 0.5)}{(p_a - p_b)}$$

and from the equation $x = (x_a + m)$ we derive the value x of the comparison stimulus for which $p = 0.5$. The results of these calculations of the intervals of uncertainty for the various subjects appear on page 56. The position of this interval with reference to a comparison stimulus of 150, the position of objective equality in our series, is of interest as evidencing a tendency in different subjects to under- or over-estimate the comparison stimulus. The values of the stimulus corresponding to the middle point of this interval are therefore given in the table. It will be noted that the values range from 148.80 (Subject II) to 152.09 (Subject III) but there seems to be no uniform tendency to under- or over-estimation since the mean value for all twelve subjects is 150.24, six being above and six below the mean. Urban's subjects judged a series of weights in comparison with a standard weight of 100 grams. Averaging similar computations for his seven subjects, we find that the middle point of the interval of uncertainty corresponds to a comparison weight of 97.56 grams, showing a decided tendency to overestimate the second (comparison) weight lifted.

GRAPHS

A study of the course of the psychometic functions for the three classes of judgments between the limits of the extremes of our series of comparison stimuli is facilitated by the use of graphs to represent these values. These were plotted and appear on pages 43 to 48. It will be seen that the values of the comparison lines 144, 146, etc., are laid off on the x-axis, and on the y-axis we have a scale of percentages from zero on the x-axis to the upper margin which represents the unit. The ordinates for $x = 144$, $x = 146$, etc., therefore represent the relative frequencies of the three classes of judgments made upon these comparison stimuli. The general type of these curves is characteristic of the psychometric functions when the comparison stimuli used vary but little from the standard stimulus. If a subject's judgments correspond in every case to the objective

relations between the comparison stimuli and the comparison stimulus, the psychometric function for shorter judgments would have the value 1 for all comparison stimuli less than the standard, would drop to zero for a comparison stimulus equal to the standard and would remain at zero for all greater values of the comparison stimulus. Conversely, the psychometric function for longer judgments would have the value zero for all comparison stimuli less than or equal to the standard and would be equal to 1 for all greater comparison stimuli. The probability of an equality judgment would be equal to 1 for a comparison stimulus equal to the standard but for all greater or less comparison stimuli would be equal to zero. This condition of affairs would be approximated if judgments were made on comparison stimuli differing from each other by an amount greater than the absolute threshold of difference: that is, a difference that was always perceived. Where the difference between the comparison stimuli themselves and between the comparison stimuli and the standard are small, the course of the psychometric functions will be much as in these charts. The psychometric function for shorter judgments begins with large values for comparison stimuli less than the standard and rapidly approaches zero for comparison stimuli at the other end of our series. That for longer judgments begins with small values for comparison stimuli shorter than the standard and approaches 1 for values greater than the standard. The psychometric function for equality judgments is represented by a graph which is more or less arched, reaching its maximum for a comparison stimulus nearly equal to the standard and assuming values nearly equal to zero at either end of our series.

It will appear, as previously noticed, that the interpolated values of the psychometric functions for comparison stimuli 145 and 155 are in some cases less than zero and in others greater than 1. Within the limits 146 and 154 for the comparison stimuli of our series there is an approximation of a type: that is, although there are marked differences in the courses of the graphs for different subjects, there is also a marked similarity in the general trend of the curves. This similarity is still more marked if we consider

only the functions for shorter and longer judgments and if we center our attention upon the part of the charts where these two cross one another rather than along the ordinate where $x = 150$. The greatest differences are noticed in the graphs for equality judgments and in the lengths and positions of the different intervals of uncertainty.

The interval of uncertainty may be found in the following manner. A straight line is drawn parallel to the base-line or x-axis of one of these charts midway between the upper and lower margins. Since $p = 0.5$ for all points on this line, the intercept lying between the points of intersection with the graphs for shorter and longer judgments respectively is the interval of uncertainty. The points of intersection, by reference to the x-axis, give the values of the thresholds in the direction of increase and of decrease.

As for the equality judgments, it is evident that there is great diversity among subjects both in the number of such judgments (shown roughly by the area enclosed below the equality graph) and also in their distribution (shown by the course of the graph, the position of its crest, etc.). It will be noticed that in only a single case (Subject XII) does the function for equality judgments exceed 0.5. For three of the subjects, however (Subject VI on 148, Subject VII on 151 and 152, and Subject XI on 150, 151, and 152) certain values of the equality function are greater than those for either longer or shorter judgments on the same comparison pairs so that, although the probability that a judgment "equal" will be given does not exceed the probability that such a judgment will not be given, there are still "odds" in favor of an equality judgment as against either a longer or a shorter judgment.

THE METHOD OF JUST PERCEPTIBLE DIFFERENCES

Many mental states may be compared quantitatively. A certain sound-sensation is more intense than another but less intense

than a third. Two different reds do not necessarily produce the same sensations of either color or brightness. One weight "feels" heavier than another. Since in all these cases the objective stimuli also vary, our everyday language for describing such mental states hopelessly confuses their subjective relations with the objective relations of the stimuli. Thus to say that "one sound is louder than another" may refer to either sensation or stimulus or to both. If we ask whether the ultra-violet rays of the spectrum are light or not, we find that we must first define whether by light we mean a form of wave-motion having certain limits of wave-length, or whether we mean a form of wave motion capable of exciting the retina. Not only may mental states such as sensations, space-perceptions, etc., be compared when it is easy to recognize that they are different, but it becomes interesting to compare two which are so nearly alike as to be almost indistinguishable since, though we recognize that they are not the same, we fail to see in what direction the difference lies. Again we may perceive no difference at all: that is, if they are subjectively equal. This does not mean that the stimuli immediately producing these subjectively equal mental states are themselves necessarily equal or nearly so. Contrast, fatigue, association, etc., may be factors in the resulting mental state so that two stimuli which are objectively equal may produce sensations or perceptions which are quite different or two very different stimuli may result in mental states which are subjectively equal. Any of the various kinds of space illusion in the comparison of the lengths of two straight lines is an example in point. Two lines may be exactly equal and yet because one is drawn in a vertical position it may "seem" considerably longer than the other drawn in a horizontal position. On the other hand, we may either shorten this vertical line or lengthen the other until they seem to be the same length.

When an observer fails to perceive any difference between two such mental states, he gives the judgment "equal" although from the difference in physical stimulation we might infer a slight difference in the mental states produced. This non-perceptible difference in sensation seems like a contradiction of terms and

yet may be explained in the following way.[1] Suppose a graded series of stimuli R_1, R_2, R_3, - - R_n so little different from one another that the sensation produced by R_2 could not be distinguished from that produced by R_1, nor that by R_3 from that by R_2, and so on, and still the sensation produced by R_7 might be easily distinguished from that given by R_1. Moreover, though a subject judging such a series would judge $R_1 = R_2$, $R_2 = R_3$, $R_3 = R_4$, - - - - - $R_{n-1} = R_n$, this is in no sense the same as judging $R_1 = R_n$. Whether we accept this reasoning as valid or not, and whether a non-perceptible difference in sensation is a contradiction of terms or not, it is self-evident that there is a non-perceptible difference in stimulation and probably in cerebral excitation.

It is the purpose of the point of view in experimentation which has come to be called *The Method of Just Perceptible Differences* to study the range of stimulation within which a subject "perceives no difference" in the resulting sensations and beyond which a difference is "perceptible". Given a series of stimuli as above, R_1, R_2, etc., producing sensations or mental states S_1, S_2, - - - - - S_n such that to the observer S_1 is "indistinguishable" from S_2, S_2 from S_3, etc., we may explore this range, we are told, in the following manner. Since S_7 is indistinguishable from S_6 and S_8, we may compare it further with S_5 and S_9, S_4 and S_{10}, etc. We shall find all sensations, we are further told, within a certain range, say S_5 to S_{10}, indistinguishable from our standard S_7. We then consider R_5 and R_{10}, the stimuli corresponding to the limits of this interval, as significant for our experiment and name R_5 the *just imperceptible negative difference* and R_{10} the *just imperceptible positive difference.* We may then compare S_7 with sensations outside of this range. Beginning, for example, with S_1, S_2, etc., we find S_4 the last to be less than S_7, and beginning higher in our series and coming back-

[1] This argument is quoted from Prof. Karl Stumpf's Tonpsychologie by G. F. Stout in his Manual of Psychology (1899), pp. 120 *et seq.* It seems to the writer that "mere sensation", "sensation", "sense-experience", and "sensory element" are unnecessary elaborations of *sensation* as this term is usually used.

ward, we may find S_{12} to be the last greater than S_7, and we have two more significant values,—R_4, the *just perceptible negative difference*, and R_{12}, *the just perceptible positive difference*. The mean of R_4 and R_5 is the *threshold in the direction of decrease*, and the mean of R_{10} and R_{12} is the *threshold in the direction of increase*. The difference between these two means gives an *interval of uncertainty or difference threshold*. All sensations caused by stimuli lying between

$$\frac{R_4 + R_5}{2} \text{ and } \frac{R_{10} + R_{12}}{2}$$

are therefore presumably "indistinguishable" from S_7.

But this "method" is not so simple as we are asked to believe. What do we mean when we say that S_7 and S_8 are "indistinguishable"? Do we mean *every* time that R_7 and R_8 are judged by our subject; or at a single trial; or in a majority of instances? For though R_7 and R_8 remain constant, S_7 and S_8 do not necessarily do so, owing to variation in attention, fatigue, practice, expectation, etc. We may vary expectation, for example, by presenting to a subject R_7 with R_1, R_2, and R_3 in serial order at one time and at another in a mixed order unknown to the subject (*method of irregular variation*). Time and space errors enter into these mental states also. Even where sources of variation are eliminated as far as possible or where every effort is made to allow for them, we still find variations in our results. Two comparison stimuli are not always judged in the same way. We have at different times $S_7 < S_8$, $S_7 = S_8$, and $S_7 > S_8$. When this difficulty arises, we may proceed to make a great number of observations on these threshold values but they are not mathematical constants and we must define them. We may define the threshold in the direction of increase as the least value of the comparison stimulus which is *always* judged greater than our standard. This would be an absolute standard but would not help us in studying our subject's sensitivity in the discrimination of small differences. Again we may consider each observation upon one of these four "differences" as equally valid with any other and take the mean

values of the four as determinations of the "just perceptible" and "just imperceptible differences". This allows for any of the variations mentioned above in that they will at times be operative in one direction and again in another. It does not make allowance for disturbing factors which always act in the same direction. It is this treatment of results that is usually employed in the method of just perceptible differences. In this discussion the thresholds and intervals of uncertainty, or difference thresholds, where this definition is applied will be spoken of as "empirical" or "observed" thresholds, etc., by the method of just perceptible differences.

If we set aside the idea of an "absolute threshold" we may arbitrarily define the threshold in the direction of increase as the least value of the comparison stimulus which is judged greater 50, 60, or 75 per cent of the time. This has the advantage of giving us a definition as soon as we may agree upon any particular percentage. It also serves as a connecting link between the error and gradation methods.

That these various questions arise will make it plain that *the method of just perceptible differences* is not a form of experimental procedure.[2] Laboratory methods may be varied considerably as long as they furnish data upon the subjective relations which hold between S_7, the result of our standard stimulus R_7. and other mental states not very different from this which are themselves the results of our comparison series of stimuli, R_1, R_2, etc. The *method of just perceptible differences* is rather a psychophysical point of view which determines the treatment of experimental data. It may make its claim on any of the "gradation methods" to furnish these data, but the laboratory procedure will depend upon the nature of the sense-organ or other body structures involved, the limitations of the apparatus used, and many other technical and practical considerations.

It will be observed that judgments upon a series of comparison stimuli, some greater and some less than the standard stimulus, arranged in varying order such as we find in this experiment, lend

[2] E. B. Holt, The Classification of Psychophysic Methods; *Psychological Review*, Vol. XI, 1904.

themselves readily to treatment by the method of just perceptible differences. Suppose that a series of our comparison stimuli were judged as follows:

156 "longer"
148 "equal"
146 "shorter"
152 "equal"
144 "shorter"
154 "longer"
150 "longer"

We may then proceed to make an empirical determination of the values of the thresholds in the following manner. We find 150 the shortest comparison stimulus which is judged longer. This is the just perceptible positive difference. We find 152 the longest comparison stimulus on which the judgment "longer" is not given. This is the just imperceptible positive difference. The threshold in the direction of increase is therefore the mean of these two values, or 151. We find further that 146 is the longest comparison stimulus which is judged shorter (the just perceptible negative difference) and that 148 is the shortest comparison stimulus on which the judgment "shorter" is not given (the just imperceptible negative difference). The threshold in the direction of decrease is therefore 147. Subtracting 147 from 151, we have the difference threshold or interval of uncertainty equal to 4. This empirical determination of the difference threshold becomes more trustworthy if we average the results of a large number of series of judgments upon our different comparison pairs.

Since the relative frequency of a certain judgment upon any comparison stimulus may be considered the probability that such a judgment will be given, we may again represent this probability by p and the probability that this judgment will not be given will be $(1 - p)$. Given $p_1, p_2, p_3, \text{- - - -} p_7$ as the probabilities of a longer judgment on comparison stimuli $x_1, x_2, x_3, \text{- - - -} x_7$, we have as probabilities that an equal or shorter judgment will be given on the same comparison stimuli $(1 - p_1)$, $(1 - p_2)$, - - - - $(1 - p_7)$. We may then set out to calculate

the probability P_n that any comparison stimulus x_n will be the shortest to be judged longer. Now P_1 is obviously equal to p_1, and the condition that x_2 may be the shortest comparison stimulus judged longer involves the probability that x_1 shall not be so judged and that x_2 shall. Thus we have

$$P_1 = p_1$$
$$P_2 = (1 - p_1)\ p_2$$
$$P_3 = (1 - p_1)\ (1 - p_2)\ p_3$$
$$- - - - - - - - - - - - - -$$
$$P_7 = (1 - p_1)\ (1 - p_2) - - - - - (1 - p_6)\ p_7$$

If P_0 represent the probability that no comparison stimulus of the series x_1, x_2, - - - - - x_7 be judged longer, we have

$$P_0 = (1 - p_1)\ (1 - p_2) - - - - - (1 - p_7).$$

Since we are dealing with mutually exclusive phenomena, in the fact that $P_1 + P_2 + P_3 + - - - - - - + P_7 + P_0 = 1$ we have a convenient check in these calculations. Having found P_1, P_2, etc., and forming the products x_1P_1, x_2P_2, etc., we have in Σx_k the mathematical expectation of the shortest stimulus to be judged longer,—that is, a theoretical determination of the just perceptible positive difference.

In like manner, if P'_n represent the probability that x_n will be the longest comparison stimulus not judged longer, we begin at the other end of our series and have

$$P'_7 = (1 - p_7)$$
$$P'_6 = p_7\ (1 - p_6)$$
$$P'_5 = p_7p_6\ (1 - p_5)$$
$$- - - - - - - - - -$$
$$P'_1 = p_7p_6 - \ \ - p_2\ (1 - p_1)$$

Here again we have a check on our calculations in that $P'_7 + P'_6 + - - + P'_0 = 1$ where $P'_0 = p_7\ p_6 - - - - p_1$ and is the probability that all the comparison stimuli will be judged longer. Having found P'_7, etc., we form the products

$x_7\ P'_7$, $x_6\ P'_6$, etc., and have for the just perceptible positive difference the value $\Sigma x_k\ P'_k$.

We may compute the negative differences in a similar manner. If q_1, q_2, q_7, - - - - q_7 are the relative frequencies or probabilities of the comparison stimuli x_1, x_2, - - - - - x_7 being judged shorter, and if Q_n repesent the probability that a comparison stimulus x_n will be the longest on which the judgment "shorter" is given, we have

$$
\begin{aligned}
Q_7 &= q_7 \\
Q_6 &= (1 - q_7)\ q_6 \\
Q_5 &= (1 - q_7)\ (1 - q_6)\ q_5 \\
&- - - - - - - - - - - - - \\
Q_1 &= (1 - q_7)\ (1 - q_6)\ - - - - -\ (1 - q_2)\ q_1
\end{aligned}
$$

Then Q_0 (the probability that no comparison line will be judged shorter) $= (1 - q_7)\ (1 - q_6)\ - - - - - - -\ (1 - q_1)$. As before, $Q_7 + Q_6 + - - - - + Q_0 = 1$. Forming the products $x_7\ Q_7$, $x_6\ Q_6$, etc., we have in $\Sigma x_k\ Q_k$ the mathematical expectation of the lonegst comparison stimulus to be judged shorter,—that is, the theoretical value of the just perceptible negative difference. Also, if Q'_n represent the probability that a comparison stimulus x_n will be the shortest on which the judgment "shorter" is not given, we have

$$
\begin{aligned}
Q'_1 &= (1 - q_1) \\
Q'_2 &= q_1\ (1 - q_2) \\
Q'_3 &= q_1\ q_2\ (1 - q_3) \\
&- - - - - - - - - - - - - \\
Q'_7 &= q_1\ q_2\ - - - - -\ q_6\ (1 - q_7) \\
\text{and}\quad Q'_0 &= q_1\ q_2\ - - - - -\ q_6\ q_7.
\end{aligned}
$$

That $Q'_1 + Q'_2 + - - - - - + Q'_7 + Q'_0 = 1$ serves as a check as in the previous calculations and in $\Sigma x_k\ Q'_k$ we have a theoretical value of the just imperceptible negative difference.

By taking the difference between the mean of the just percep-

tible and just imperceptible positive differences and the mean of the just perceptible and just imperceptible negative differences we get a theoretical value of the interval of uncertainty, or difference threshold. This is equal in the case of each subject to

$$\frac{\Sigma x_k P_k + \Sigma x_k P'_k}{2} - \frac{\Sigma x_k Q_k + \Sigma x_k Q'_k}{2}$$

These calculated values of the thresholds and of the intervals of uncertainty for each of the twelve subjects are placed in Table VIII (p. 56) in parallel columns with the observed or empirical values by the method of just perceptible differences and with the values obtained by interpolation. It is interesting to compare the results.

If we arrange our twelve subjects according to the magnitude of the interval of uncertainty as obtained by interpolation and in a parallel column we arrange them according to the observed value of the interval by the method of just perceptible differences, we have the following result.

Comparison of intervals of uncertainty (1) by interpolation and (2) by the method of just perceptible differences.

(1)		(2)	
VI	8.52	VI	4.15
VII	5.41	XII	3.94
XI	3.96	XI	3.67
III	3.09	VII	3.53
XII	3.07	II	2.82
II	2.93	IX	2.76
V	2.57	I	2.72
IX	2.47	V	1.71
I	2.24	III	1.55
IV	1.96	IV	1.23
X	0.92	X	1.05
VIII	0.36	VIII	0.51

Although the values obtained by the two methods are in some cases quite different, there is a general similarity in the two lists and the relative positions of the different subjects are much the same. If we leave subjects III and V out of account, the order of the subjects in the two lists is the same except that subjects XII and VII change places.

If we consider the middle points of these observed intervals of uncertainty, *i.e.*, the mean of the thresholds in the directions of increase and decrease respectively, as we did above (p. 23) in the case of the values by interpolation, we find a similar result. The mean for the twelve subjects is 150.05, ranging from 151.71 (Subject III) to 148.57 (Subject IX). In this case also there are six above and six below the mean value. This may confirm us in our conclusion that there was no uniform tendency to over- or under-estimation.

A comparison of the theoretical intervals of uncertainty calculated by the method of just perceptible differences with the observed or empirical values by the same method is less gratifying. As appears in the following comparative table, Subjects VI, XII, II, I, III, IV, and X maintain the same relative positions in both columns but there is little regularity in the relations of the other five.

Comparison of intervals of uncertainty by the method of just perceptible differences:

(1) calculated		(2) observed	
VI	6.67	VI	4.15
IX	3.44	XII	3.94
XII	3.38	XI	3.67
V	3.00	VII	3.53
II	2.78	II	2.82
I	2.35	IX	2.76
III	—2.17	I	2.72
XI	1.16	V	1.71
VII	0.69	III	1.55
VIII	0.62	IV	1.23
IV	0.57	X	1.05
X	—0.09	VIII	0.51

The mean of the calculated thresholds (149.00) is smaller than the observed value with practically the same mean variation among the different subjects. A study of the calculated thresholds will show that the threshold in the direction of decrease is less in every case but one (Subject VII) than the threshold by interpolation, and in every case but one (Subject XII) less than the observed threshold. Furthermore, the calculated thres-

holds in the direction of increase are in every case smaller than the thresholds obtained by either of the other two methods for the same subjects. In a few cases[3] there is striking agreement in the results of the three methods.

A study of the calculations by this theoretical method of just perceptible differences shows that, in the cases of a number of subjects whose relative frequencies for shorter judgments on comparison line 144 and for longer judgments on 156 do not exceed 0.80, the probability that no one of our comparison lines will be judged shorter or longer respectively is not inconsiderable. This suggests that, for these subjects at least, our series should have been more extended. That this lack of agreement between the calculated and observed intervals of uncertainty is largely due to large values of P_0 and Q_0, that is to probabilities that no comparison stimulus in our series would be judged longer or shorter respectively, and that these in turn are due to values of p and q which are less than 0.80 is borne out by the following comparison. We have arranged our subjects in order according to the extent to which their calculated intervals of uncertainty vary from the observed values and have placed in parallel columns the highest of the four values P_0, P'_0, Q_0, and Q'_0 for each subject, and in the third and fourth columns the values of q_1 and p_7.

Subject	Difference	Maximum P_0, etc.	q_1	p_7
III	−3.72	$P_0 = 0.064$	0.804	0.628
VII	−2.84	$P_0 = 0.047$	0.782	0.680
VI	+2.52	$Q_0 = 0.068$	0.586	0.718
XI	−2.51	$P_0 = 0.033$	0.900	0.796
V	+1.29	$Q_0 = 0.020$	0.752	0.842
X	−1.14	$P_0 = 0.019$	0.940	0.796
IX	+0.68	$Q_0 = 0.025$	0.750	0.896
IV	−0.66	$P_0 = 0.012$	0.956	0.794
XII	−0.56	$Q_0 = 0.007$	0.944	0.978
I	−0.37	$P_0 = 0.007$	0.878	0.862
VIII	+0.11	$Q_0 = 0.001$	0.900	0.948
II	−0.04	$Q_0 = 0.003$	0.860	0.848

[3] Subject II had thresholds 147.28, 147.34, and 147.41 in the direction of decrease and 150.06, 150.27, and 150.23 in the direction of increase; Subject VIII had thresholds 149.53, 149.91, and 149.73 in the direction of decrease and 150.15, 150.27, and 150.24 in the direction of increase.

It may be observed from this comparison that where p_7 is relatively small, P_0 is large and we find a calculated interval less than the observed. On the other hand, where q_1 is small, Q_0 is large and, in four out of six cases, the calculated interval is larger than the observed. It seems likely from this comparsion that there would have been closer agreement between the results of these two methods of deriving the difference threshold, or interval of uncertainty, if we had either (1) added comparison stimuli 142 and 158 at the lower and upper ends of our series respectively, or (2) had kept the number of our comparison pairs at seven but had used comparison lines 141, 144, 147, 150, 153, 156, and 159. Here the question also arises as to whether the same comparison pairs should be used for each subject or whether we should so adjust the comparison pairs for each subject that the relative frequency of shorter judgments on the shortest comparison line and of longest judgments on the longest line should exceed some arbitrary minimum, say 0.95. It is obvious that if we had used 141 and 159 as the extremes of our series of comparison lines in this experiment, the last five subjects in the above table would have had relative frequencies nearly equal to unity for shorter and longer judgments respectively on these lines. It may be remarked in this connection that in the preliminary experiments with a more extended series, when the present series was being decided upon, Subject II was used to a considerable extent. If Subjects III or VII had been used for this preliminary work, it is very likely that our series would have been too extended for the majority of observers for there is little point in having a comparison pair upon which the probability of a certain type of judgment amounts to a practical certainty.

EQUALITY CASES

If we consider each equality judgment as an observation upon the value of the stimulus giving rise to the same sensation of length as our standard stimulus, we may apply the same methods

of computation as in a series of determinations of some quantity in chemical experimentation,—the atomic weight of an element, for example. We derive the arithmetical mean of all our observations as the most probable value of the magnitude to be determined. We then proceed by the method of least squares to compute the probable error of this mean result. If we have n obversations varying from the mean by v_1, v_2, etc., we sum the squares of these variations and compute the probable error of the mean result by the formula

$$0.6745\sqrt{\frac{\Sigma v^2}{n(n-1)}}$$

It has already been noticed that the graph of the psychometric function for equality judgments has, in typical cases, more or less of the bell shape of the error curve,—particularly if we consider its course in the neighborhood of the comparison stimulus equal to the standard. The maximum value of the psychometric function for equality judgments has been computed in the following manner. If the three largest values of the probability of an equality judgment for a given subject as found in our table of relative frequencies are p_a, p_b, and p_c, corresponding to values x_a, x_b, and x_c of the comparison stimuli, and the value of the comparison stimulus corresponding to the maximum of the psychometric function for equality judgments x_m differ from our middle value x_b by d, we have, on the assumption of parabolic form,

$$d = \frac{p_a - p_c}{2(p_a + p_c - 2p_b)}, \text{ and}$$

$$x_m = x_b + d.$$

It is of course necessary to observe the sign of d as it is sometimes positive and sometimes negative. We may then find the maximum value of the psychometric function for equality judgments p_m by introducing x_m into Lagrange's formula.

It is interesting in connection with the use of error methods in psychophysics to compare the values of the maximum of the psychometric function for equality judgments with the values of

the probable error of the mean result for our different subjects. The result is what might be expected from the similarity of the course of the equality function and the error curve. It will be noticed that as the maximum of the equality function decreases (column 1), the probable error increases. This comparison is rather striking when we recall the earlier observations made upon the diversities of the number and distribution of these very equality judgments.

Subject	Maximum	Probable Error
XII	0.536	0.058
XI	0.443	0.070
VII	0.396	0.077
VI	0.394	0.078
I	0.323	0.087
III	0.285	0.092
II	0.220	0.100
IX	0.212	0.105
V	0.204	0.110
IV	0.192	0.117
X	0.111	0.150
VIII	0.069	0.175

It is to be expected from the fact that the sum of the three ordinates for the psychometric functions for any comparison stimulus is equal to one that this maximum of the equality function is to be found where the values for the shorter and longer judgments are both small,—that is, in the neighborhood of the interval of uncertainty. It will be seen by reference to the tabulation of the analysis of equality judgments (p. 55) that this value of the comparison stimulus x_m lies within the interval of uncertainty both as observed by the method of just perceptible differences and as determined by interpolation for eight of our twelve subjects (the exceptions being V, VII, VIII, and X), We note further that x_m has a mean value of 149.91 with considerable variation (M.V. = 1.68).

If, as above, we consider each of these equality judgments as determinations of a certain value x, we find that the mean of the equality judgments of each subject differs but little from 150 with an average of 149.94 for all twelve subjects and a mean

variation of 0.69. If we make allowance for the different number of this class of judgments made by each the result is much the same. We have in all 7445 equality judgments with a mean value of 149.89. These results agree with our study of shorter and longer judgments in showing that there is no marked tendency to either under- or over-estimate the comparison stimulus.

If we consider the absolute frequency of the equality judgments made upon each of our seven comparison stimuli as weights of these seven observations upon our standard stimulus we may compute the probable error of a single result from the formula

$$0.6745\sqrt{\frac{\Sigma v}{6}}$$

wher Σv^2 is the sum of the weighted squares of the variations from the mean result. These results are tabulated in column 5 of the analysis of equality judgments on page 55. It is of interest to compare them with some other test of the sensitivity of our subjects. If these are arranged according to the length of the interval of uncertainty by interpolation and according to the probable errors just derived, we have the following:

Interval of uncertainty by interpolation		Probable error of a single result	
VI	8.52	VI	32.43
VII	5.41	VII	31.16
XI	3.96	XI	26.31
III	3.09	III	25.78
XII	3.07	I	23.33
II	2.93	V	22.99
V	2.57	IX	22.34
IX	2.47	II	22.14
I	2.24	XII	20.19
IV	1.96	IV	16.08
X	0.92	X	15.91
VIII	0.36	VIII	9.62

It will be noticed that, except for Subjects I, III, and XII, these orders are the same.

The following table makes a more general comparison possi-

ble. In the first column the subjects are arranged according to the length of the interval of uncertainty by interpolation, in the second according to the length of the interval by the method of just perceptible differences, in the third according to the probable error of the mean of the equality judgments, in the fourth according to the maximum of the equality function, and in the fifth according to the probable error of a single equality determination. (Subjects III and XII are omitted.)

(1)	(2)	(3)	(4)	(5)	
VI	VI	XI	XI	VI	
VII	XI	VII	VII	VII	A
XI	VII	VI	VI	XI	
II	II	I	I	I	
V	IX	II	II	V	B
IX	I	IX	IX	IX	
I	V	V	V	II	
IV	IV	IV	IV	IV	
X	X	X	X	X	C
VIII	VIII	VIII	VIII	VIII	

This falls short of the remarkable agreement of Dr. Urban's four arrangements of his seven subjects but is confirmatory in a general way of his main conclusion "that there must exist some kind of relation between the values of the different psychometric functions, and there must also exist a relation between the probable error and the maximum value of the psychometric function of the equality judgments". It may be noticed that, if we divide the ten subjects listed above into three groups A, B, and C, putting the first three in A, the next four in B, and the last three in C, no subject from one group in any arrangement appears out of his group in any other.

It is obvious that as one increases the number of his subjects, there is less likelihood that he will find perfect agreement in different serial arrangements. He will look rather for slight variations in any one subject from an average position. The expectation of absolute uniformity in serial arrangement for a number of subjects will be less in the middle of a series than at its extremes for, if the number of subjects is increased, it is not likely that the range of values for any of these tests of

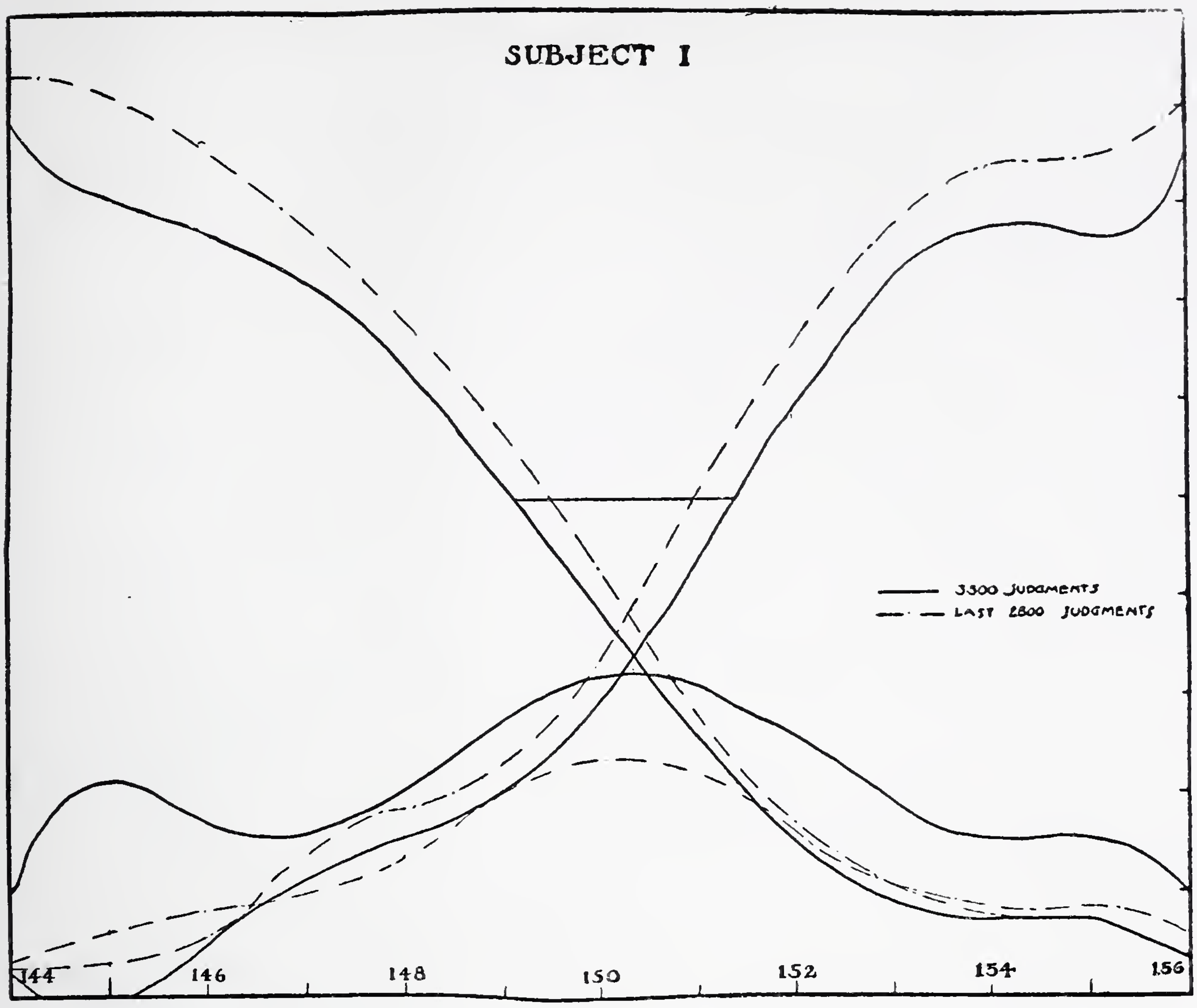
sensitivity will be greatly changed and so we may expect more additions to find place in the middle of any column than at its extremes. As these middle values approximate a mean, it requires only slight differences in them to effect a change in the serial order. This is shown in our present experiment by the fact that, if we include all twelve subjects in an arrangement like the above, in column (1) six values will lie within a range of 8.4 per cent of the difference between the first and last in the column, in column (2) five within a range of 30.5 per cent, in column (3) six within a range of 24.5 per cent, in column (4) five within a range of 25.5 per cent, and in column (5) five within a range of 19.2 per cent.

DESCRIPTION OF APPARATUS

As stated above (p. 2), the apparatus used was particularly designed to serve in this experiment. It is essentially a revolving drum (A in Fig. 2, p. 49) mounted on an axle behind a cardboard screen (D) and driven by an "intermittent gear" (B) which in turn was driven by an electric motor. The drum is a fourteen-sided prism made of light wood. On seven alternate faces are nailed thin strips of wood, 3/8″x3½″x12″. The intermediate faces are covered with white paper. Each of the seven raised faces is fitted with six small brass fasteners projecting ¼″ over the edges of the outer surface. These fasteners hold the cards, also 3½″x12″, firmly in place and yet allow the cards to be changed easily. On the end of the drum nearer the recorder these seven faces are distinguished by numbers from 1 to 7. The axles for the drum and the intermittent gear wheels are ½″ in diameter and rest in bearings mounted on a box framework of 7/8″ white pine. The drum axle is held tightly enough by its bearings so that it will not turn very freely but retains the position in which the intermittent gear leaves it. The wheel O is made fast to the drum axle. It is made with seven rests which allow the drive wheel to continue its revolution and yet leave the drum at rest. Belts such as are used for

sewing-machines connect the apparatus to a speed-reducer (S) which is connected to a small motor. The belts which revolve most rapidly are sewed together rather than held by the ordinary metal fasteners in order to prevent any clicking sounds as they pass over the wheels.

Drum, screen, speed-reducer, and motor were all set up on a large laboratory table (T) along one side of the experiment room. The subject sat behind a small table facing the screen; the recorder sat where he could see the numbers on the drum. The screen consists of a frame holding a sheet of white cardboard into which a horizontal opening (C) 2½"x10" is cut at such a height from the table that when any card (as No. 3 in Fig. 2) is brought into position, the center of the card is behind the center of the opening. This screen is high enough to shut off entirely from the subject any view of the rest of the apparatus. The margins of the cards do not appear when at rest and, except for a slight shadow, there is little break in the continuity of the background on which the comparison lines are seen.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3300 JUDGMENTS
LAST 2800 JUDGMENTS
144
146
148
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152
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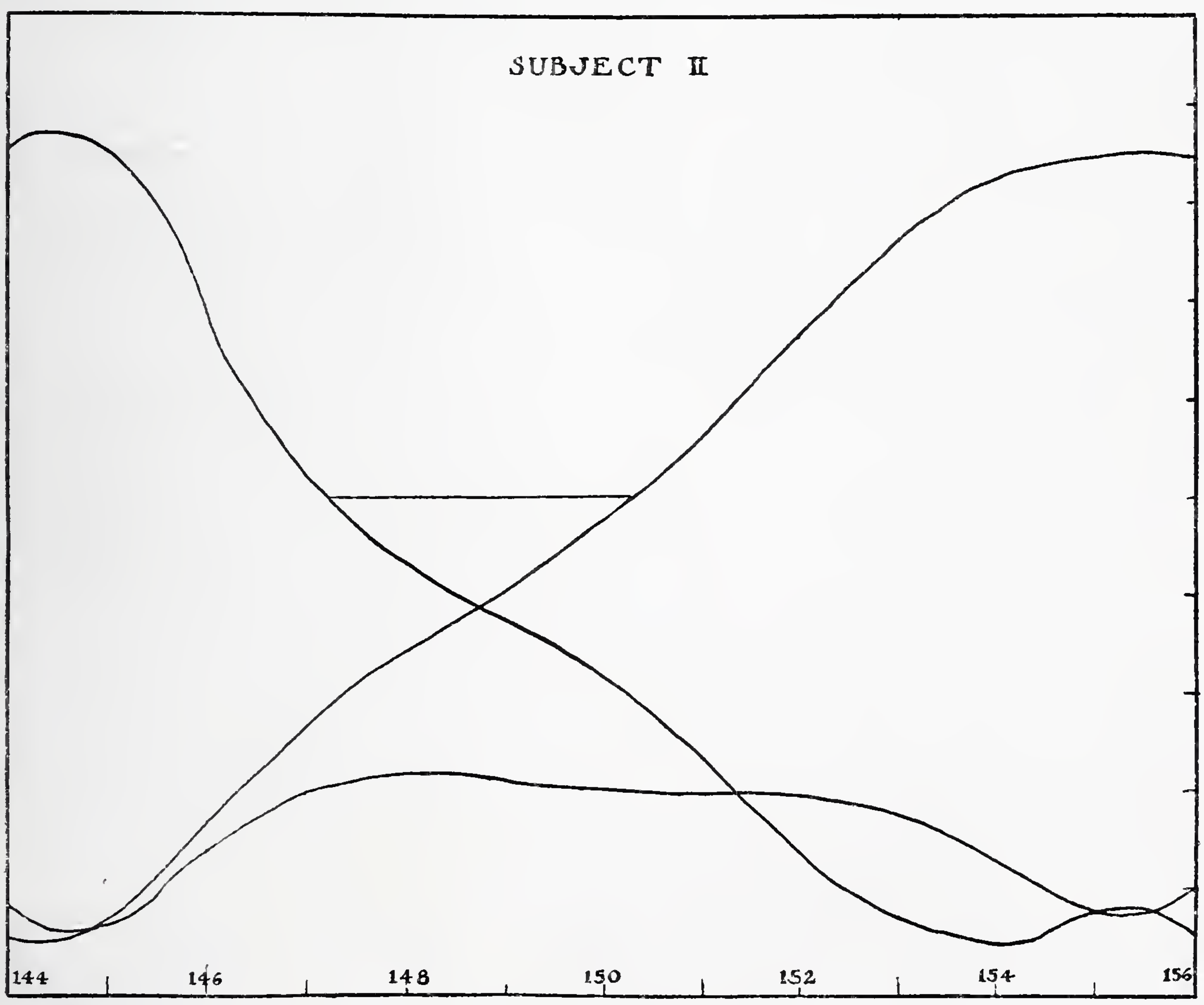
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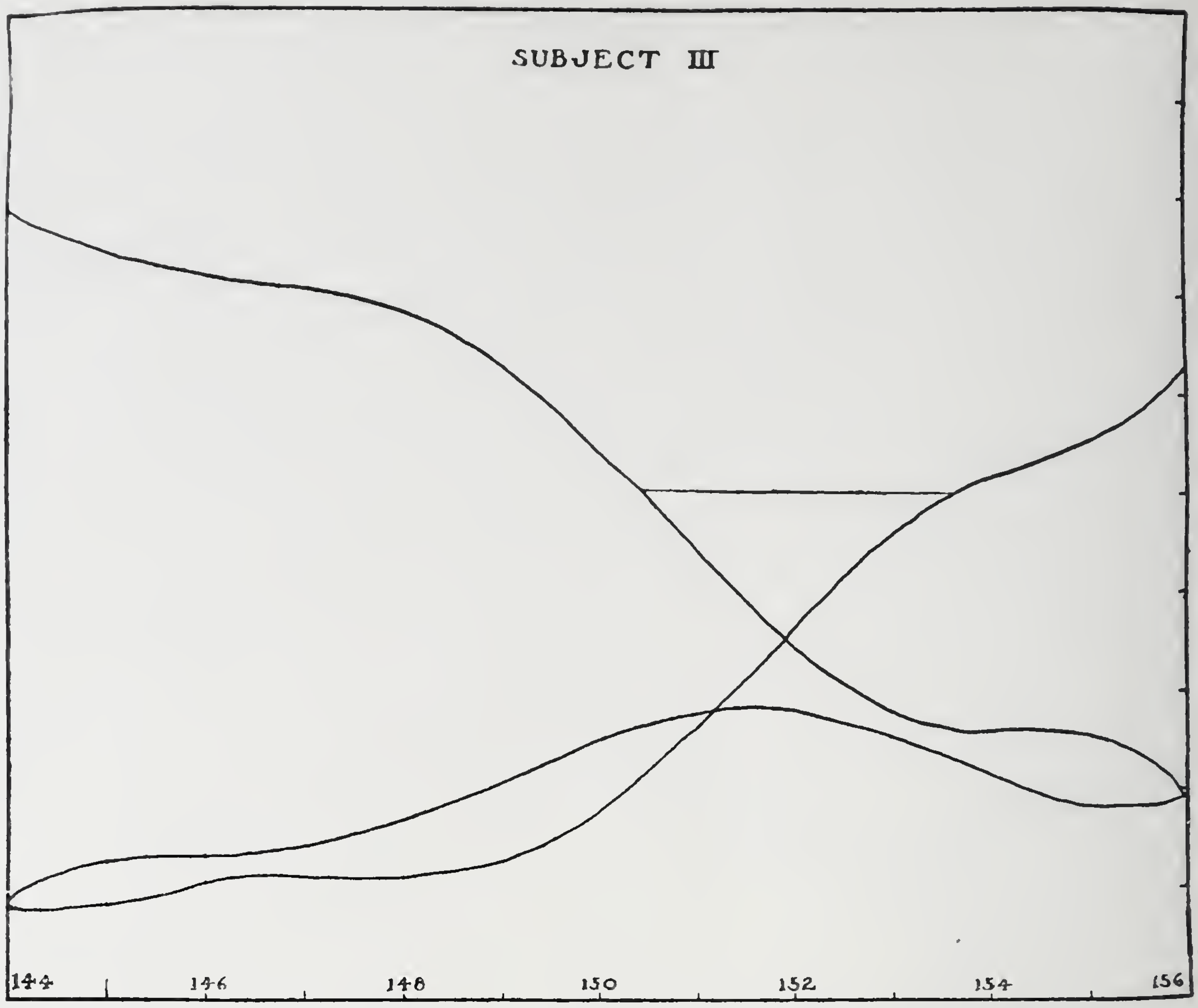
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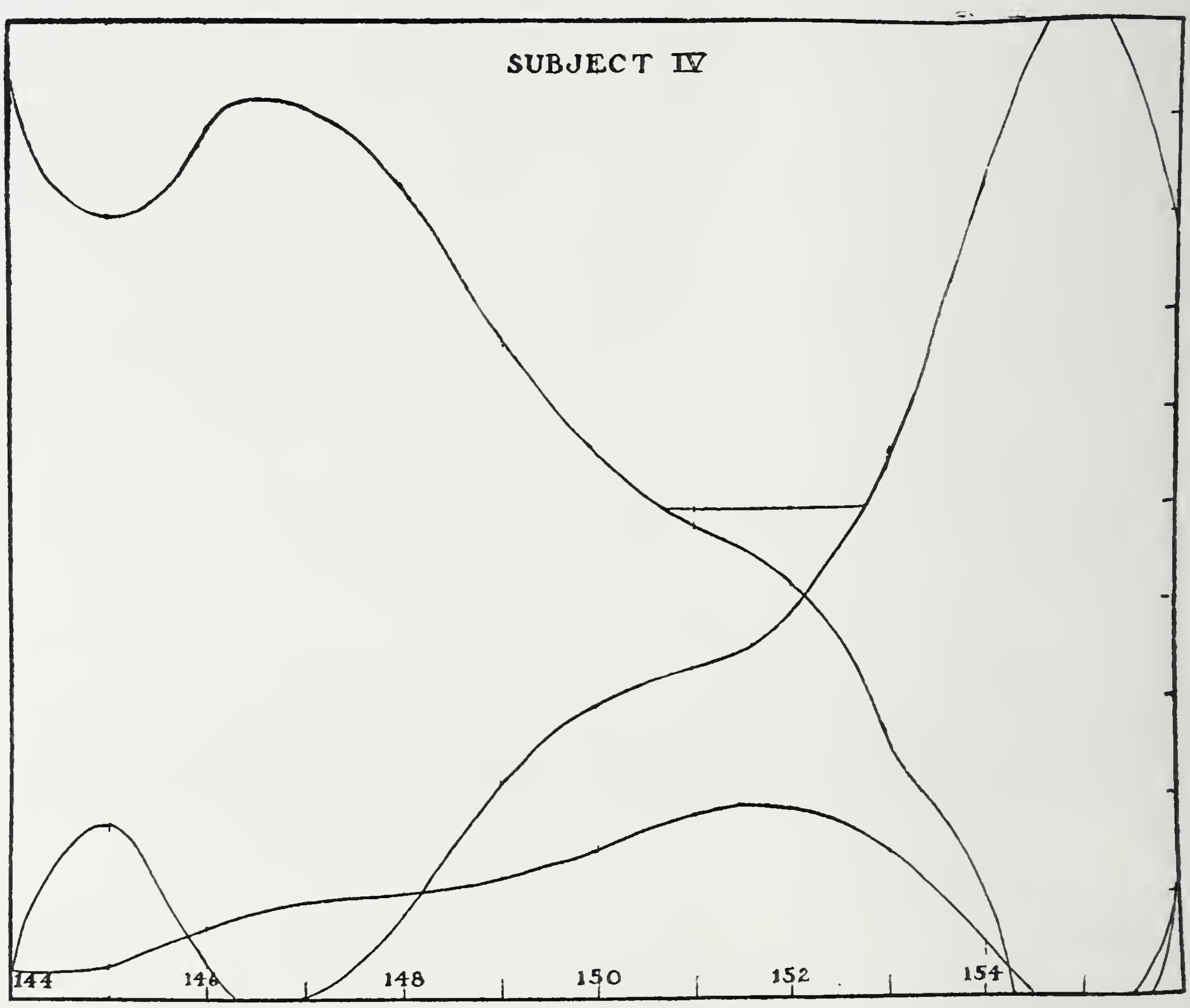
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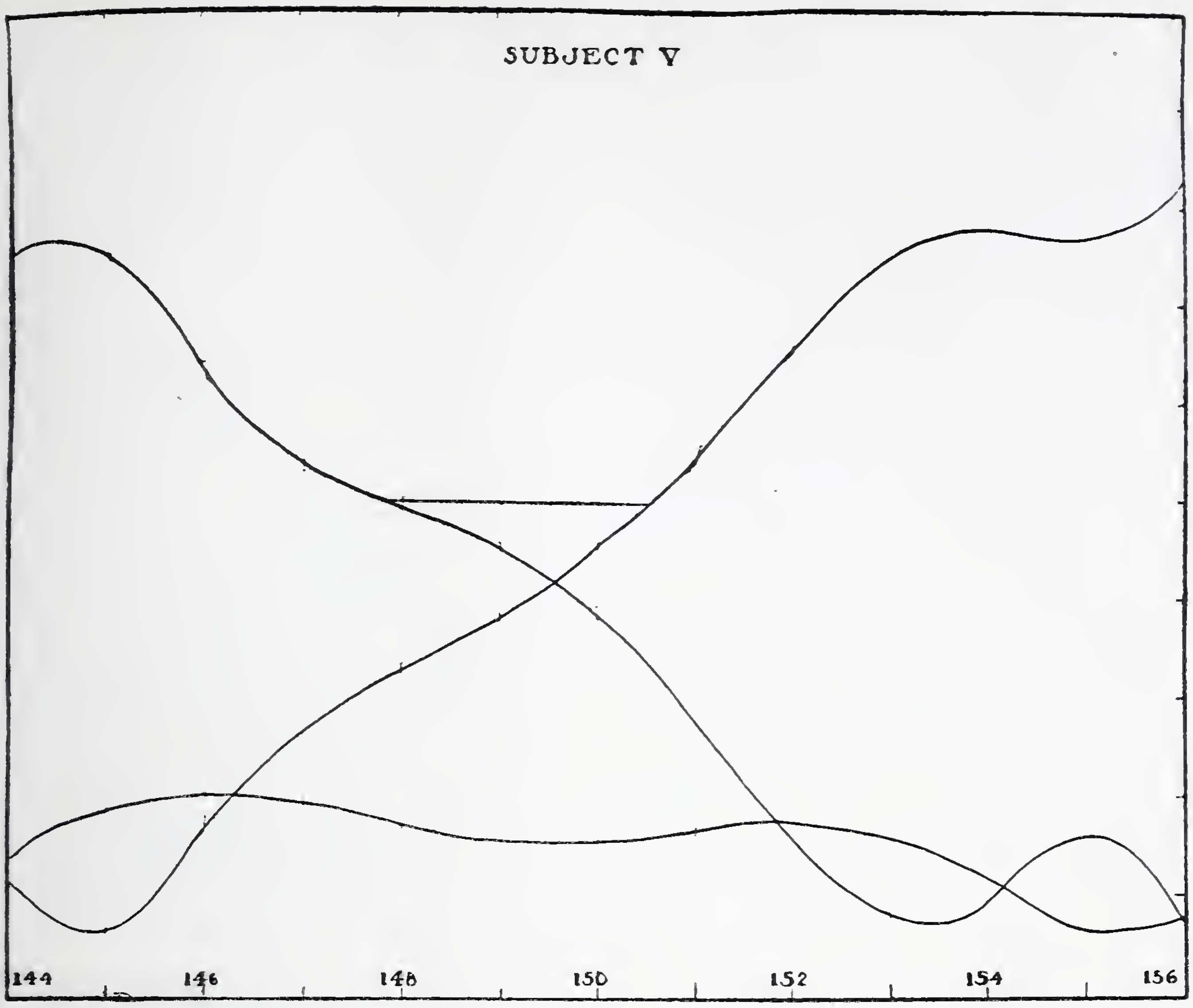
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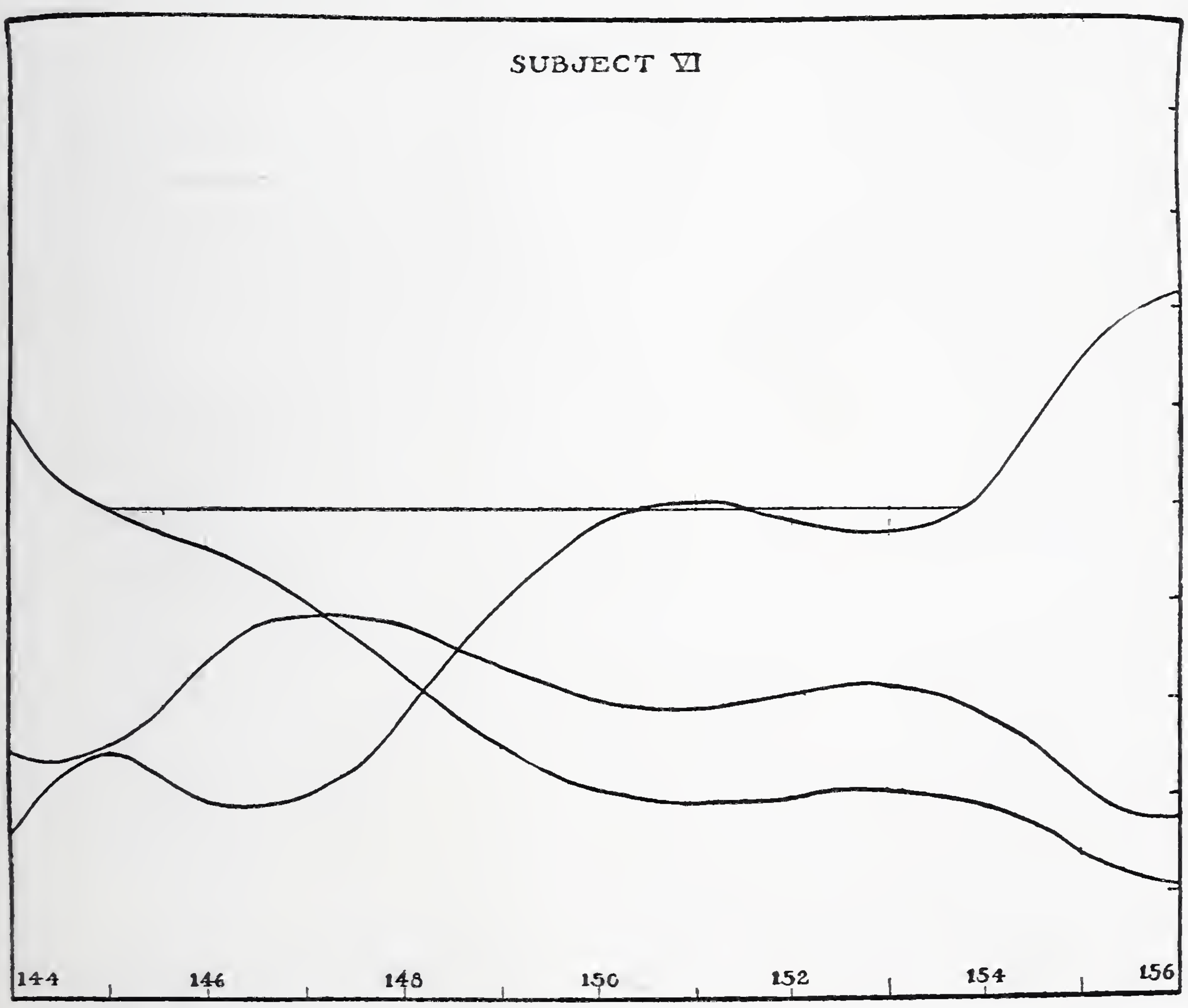
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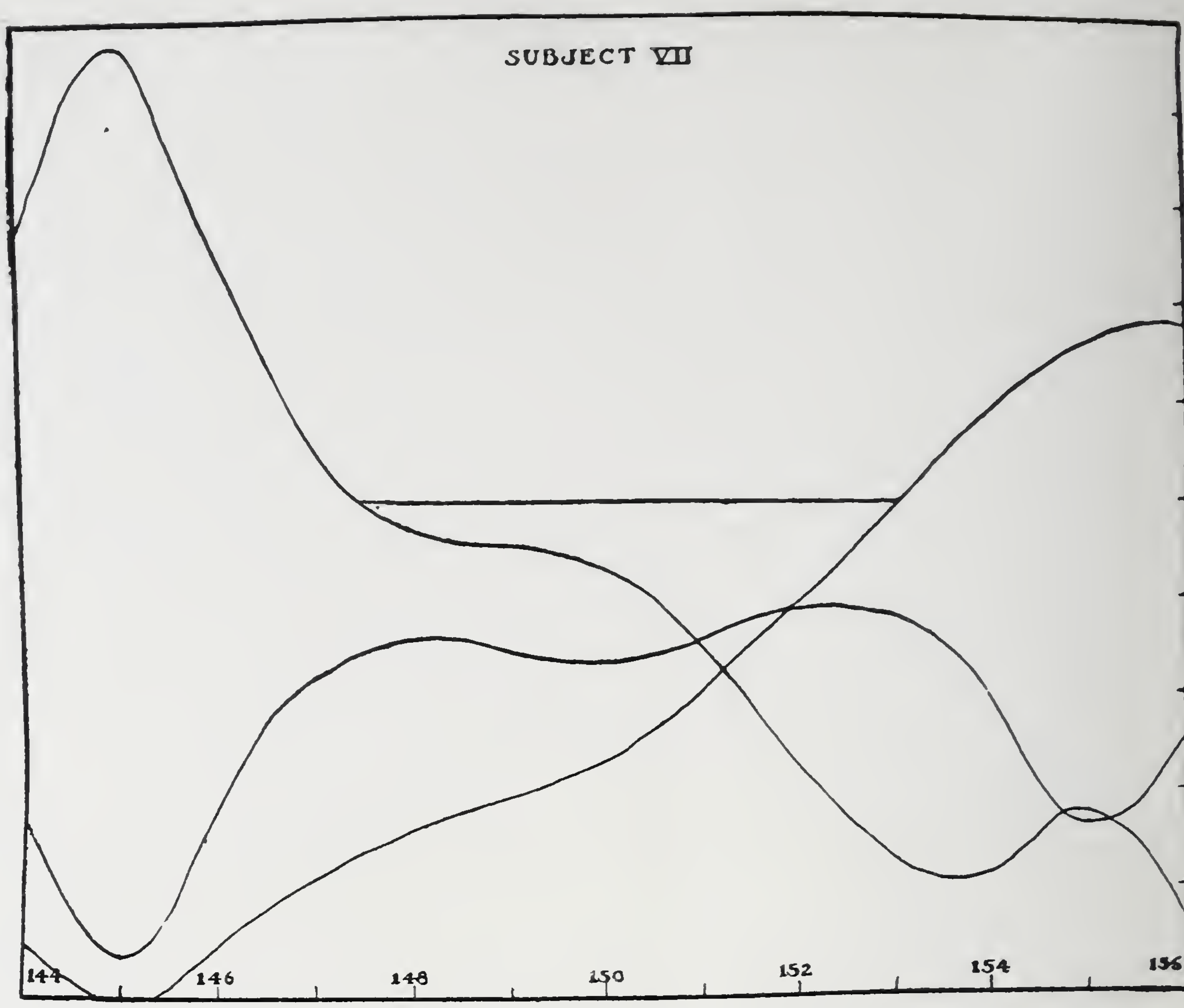
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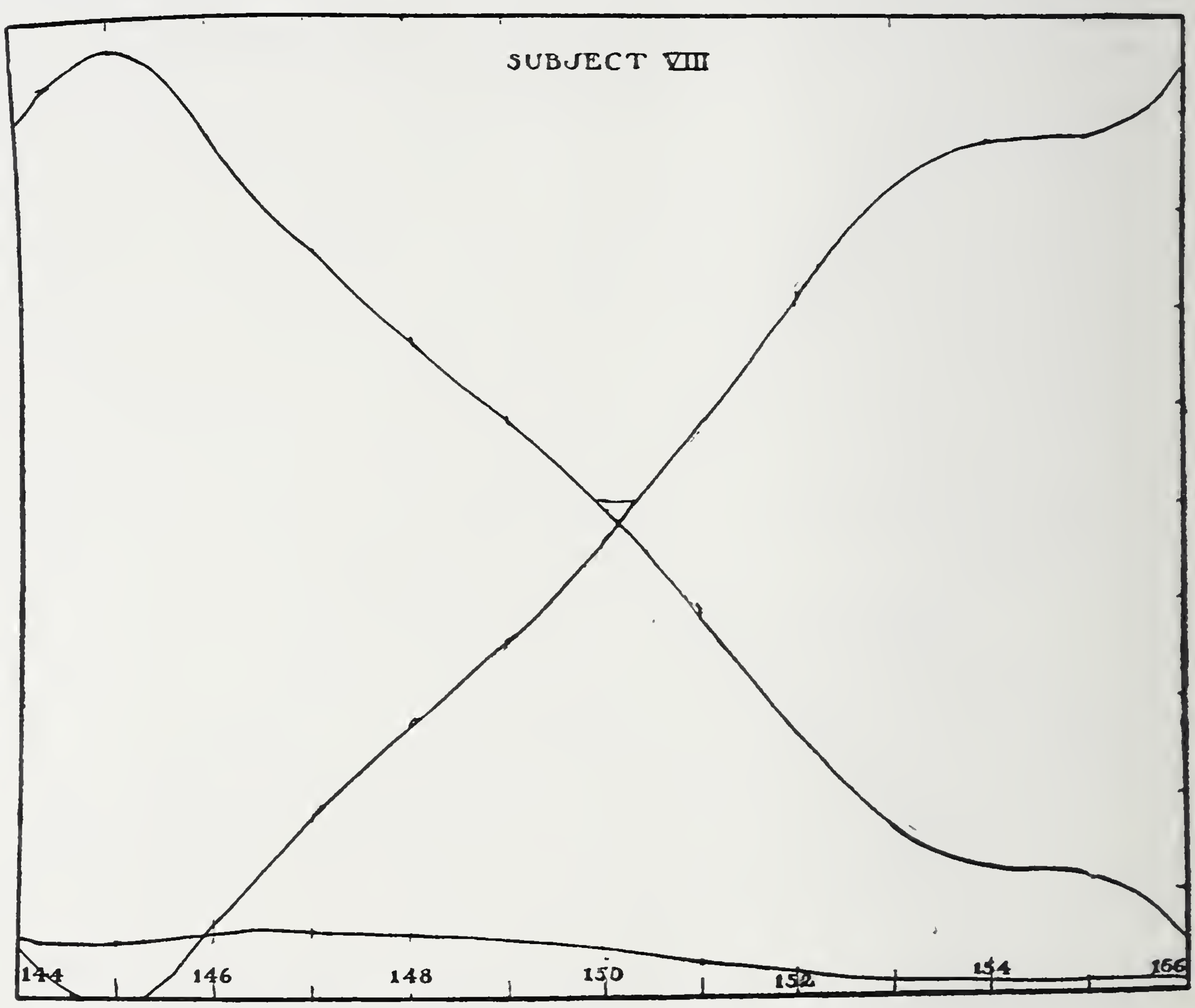
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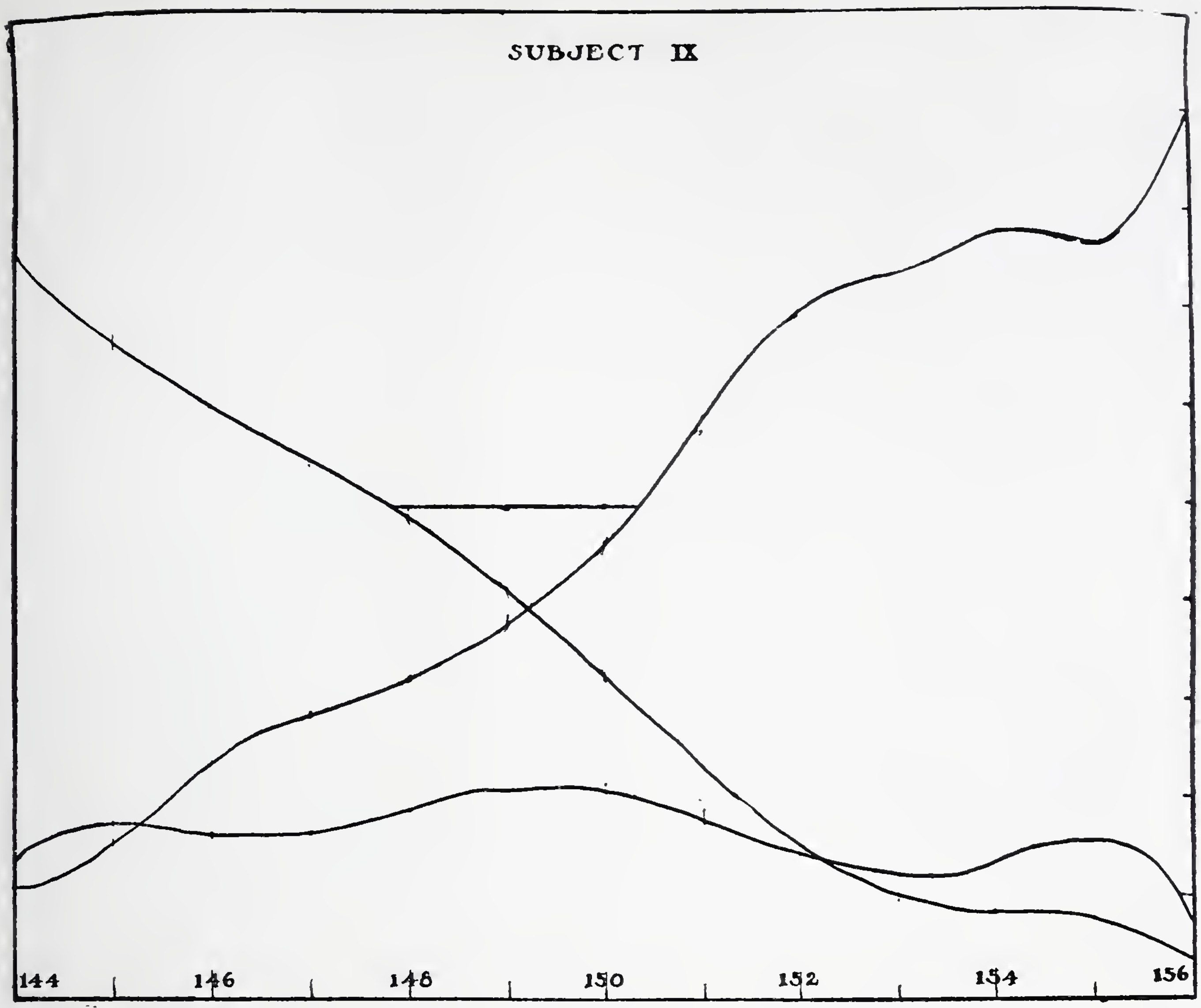
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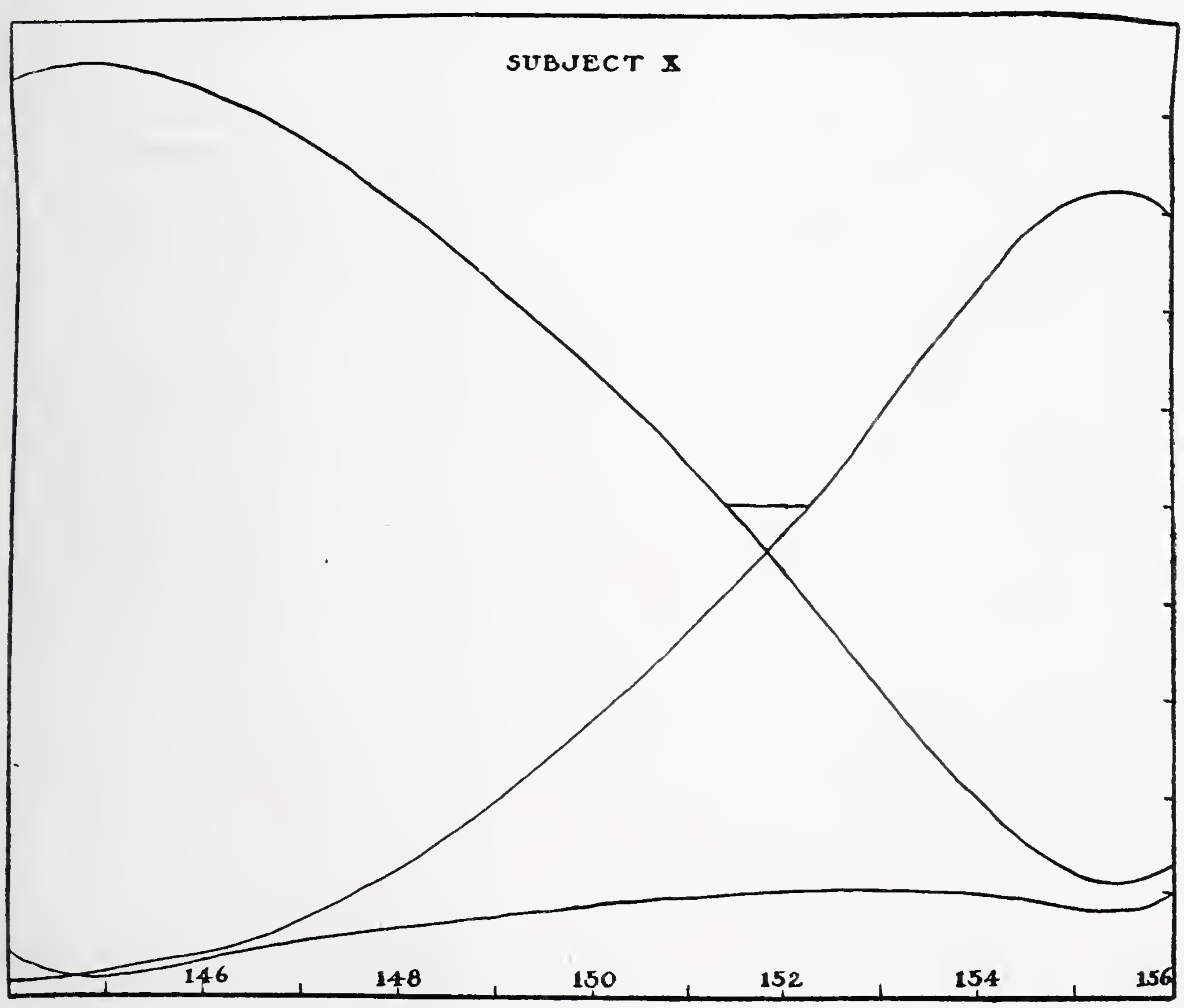
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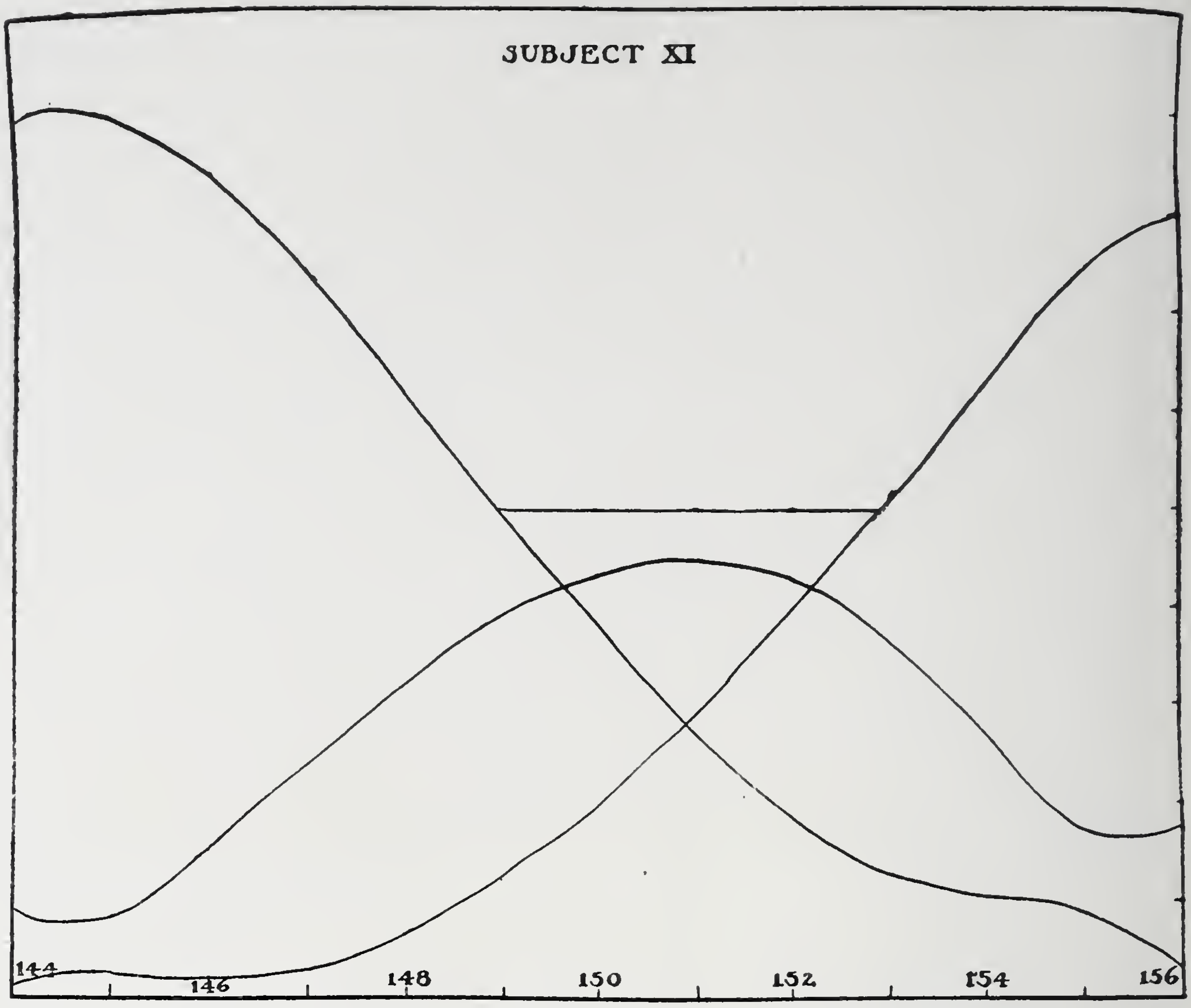
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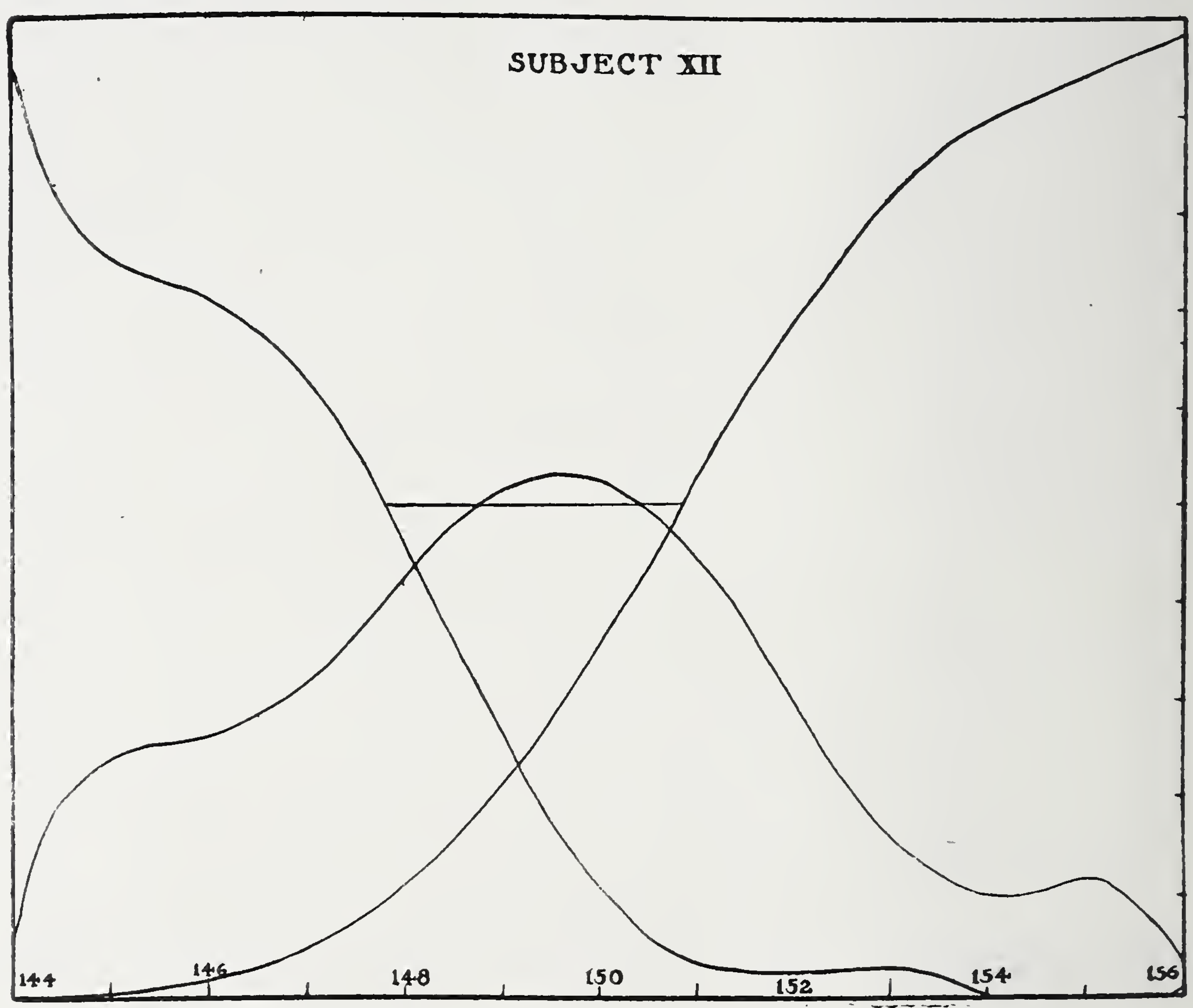
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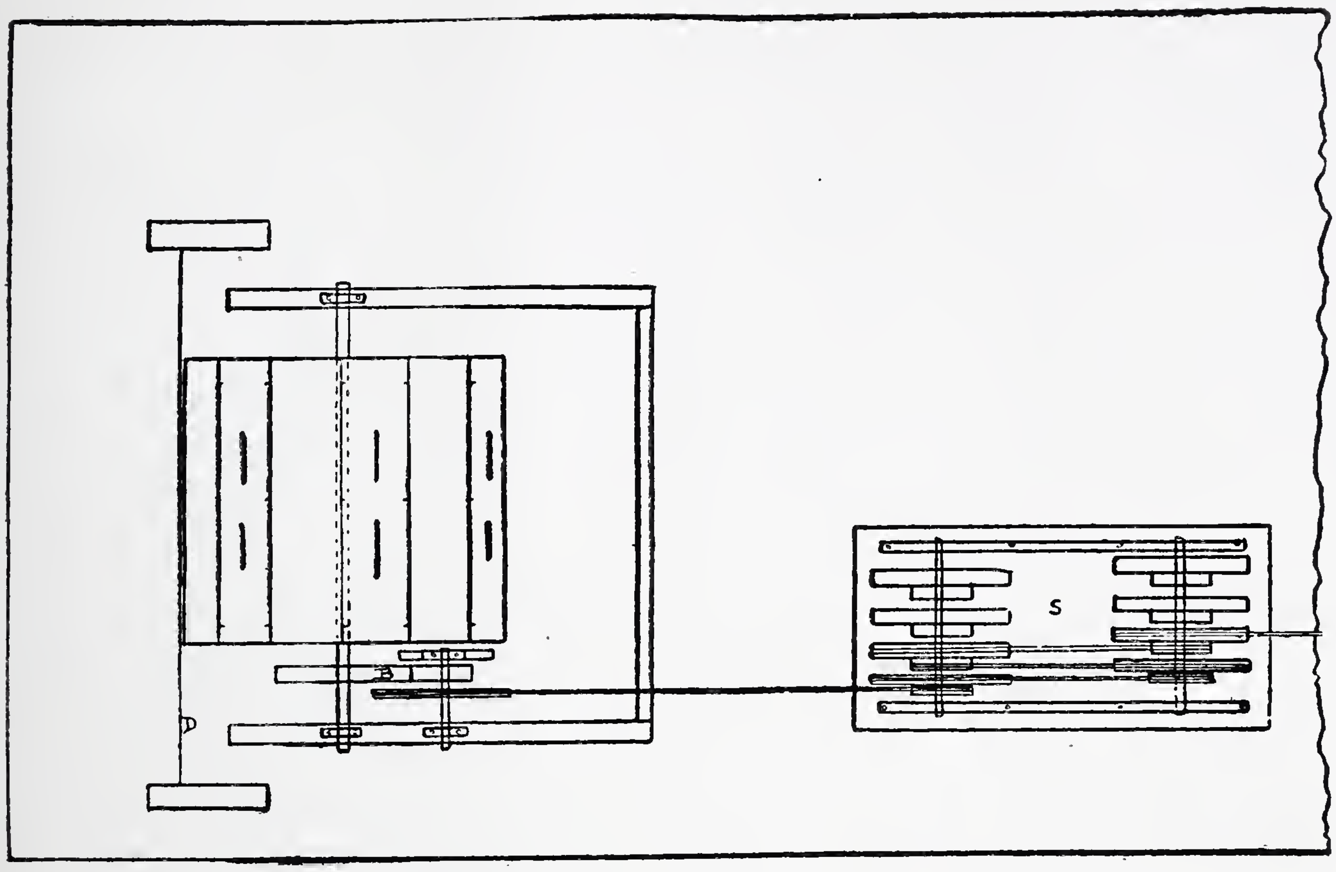

Figure 1

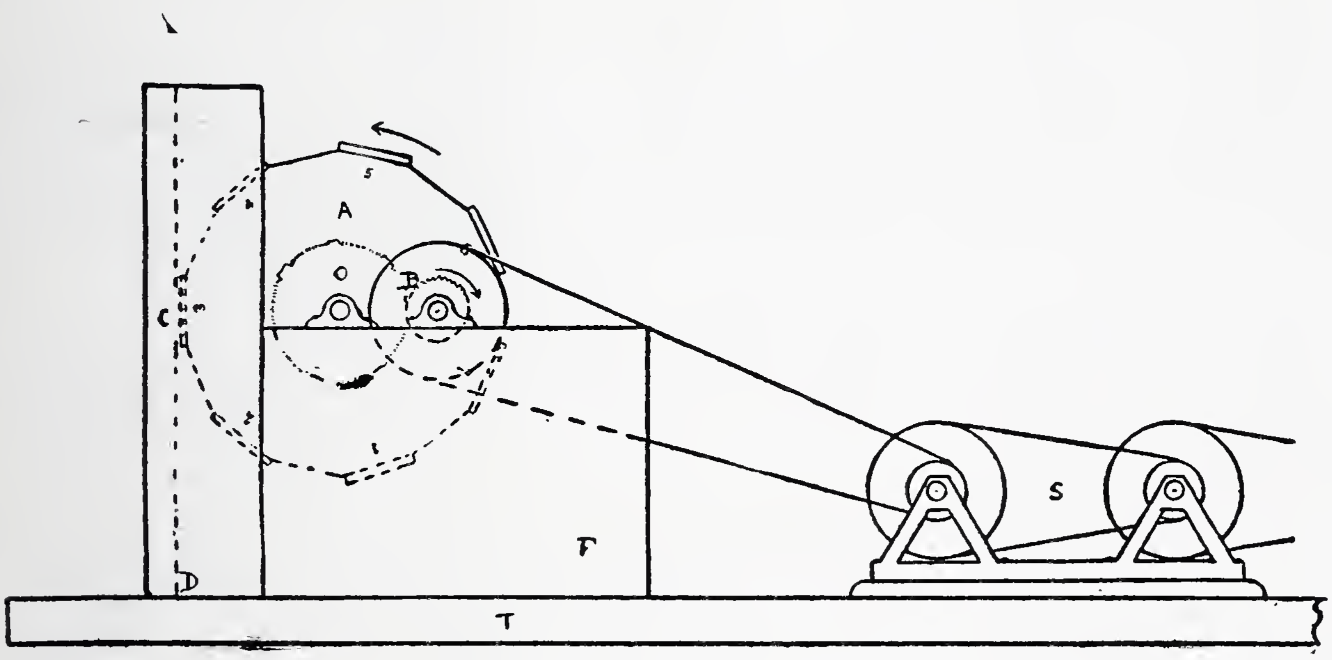

Figure 2

Table I. Absolute Frequencies of Judgments

Subject I

C. S.	144			146			148			150			152			154			156		
Series	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l
Ia	31	19	0	20	30	0	20	28	2	15	30	5	3	31	16	6	22	22	2	15	33
b	34	16	0	23	26	1	26	22	2	7	36	7	5	30	15	2	20	28	1	13	36
IIa	46	3	1	43	6	1	32	11	7	20	13	17	5	20	25	4	13	33	2	5	43
b	42	7	1	41	4	5	28	15	7	12	20	18	8	17	25	2	9	39	6	6	38
IIIa	44	3	3	39	7	4	24	10	16	18	13	19	7	5	38	2	1	47	3	2	45
b	47	0	3	40	8	2	42	3	5	22	18	10	1	5	44	2	1	47	0	1	49
IVa	47	1	2	44	0	6	41	1	8	28	5	17	16	2	32	9	1	40	3	1	46
b	49	0	1	47	3	0	40	5	5	13	19	18	14	16	20	4	8	38	0	4	46
Va	49	1	0	46	4	0	38	6	6	30	6	14	9	1	40	2	2	46	2	3	45
b	50	0	0	44	1	5	29	2	19	29	0	21	8	0	42	3	0	47	0	0	50
Σ	439	50	11	387	89	24	320	103	77	194	160	146	76	127	297	36	77	387	19	50	431

Subject II

C. S.	144			146			148			150			152			152			154		
Series	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l
Ia	46	3	1	39	8	3	26	18	6	25	21	4	12	16	22	3	16	31	3	15	32
b	46	2	2	39	10	1	32	8	10	23	15	12	5	37	8	5	7	38	4	10	36
IIa	37	12	1	38	8	4	18	19	13	10	16	24	6	6	38	4	9	37	6	10	34
b	37	11	2	37	10	3	21	17	12	15	13	22	8	13	29	3	10	37	4	13	33
IIIa	42	2	6	20	5	25	22	2	6	13	6	31	6	3	41	1	1	48	0	3	47
b	44	2	4	30	2	18	13	6	31	16	3	31	5	2	43	2	3	45	2	1	47
IVa	45	1	4	37	7	6	25	7	18	9	7	34	8	8	34	0	6	44	0	0	50
b	45	3	2	34	7	9	15	17	18	21	4	25	7	7	36	0	6	44	3	1	46
Va	42	5	3	33	5	12	23	11	16	16	13	21	7	3	40	5	3	42	0	1	49
b	46	2	2	37	9	4	24	5	21	12	4	34	5	3	42	0	2	48	0	0	50
Σ	430	43	27	344	71	85	219	110	171	160	102	238	69	98	333	23	63	414	22	54	424

Subject III

C. S.	144			146			148			150			152			154			156		
Series	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l
Ia	13	20	17	14	16	20	14	19	17	19	15	16	7	21	22	21	13	16	9	21	20
b	21	17	12	18	18	14	10	23	17	23	12	15	13	16	21	11	14	25	14	18	18
IIa	34	4	12	34	4	12	36	4	10	19	7	24	16	3	31	12	0	38	16	4	30
b	45	2	3	46	0	4	46	3	1	31	10	9	24	2	24	9	5	36	19	2	29
IIIa	47	0	3	39	5	6	38	6	6	37	7	6	25	5	20	14	6	30	14	5	31
b	50	0	0	40	7	3	40	7	3	35	13	2	14	18	18	15	9	26	3	9	38
IVa	46	4	0	43	7	0	40	6	4	22	21	7	18	21	11	10	19	21	5	10	35
b	50	0	0	46	4	0	44	4	2	29	16	5	23	19	8	10	18	22	4	10	36
Va	47	3	0	46	4	0	42	8	0	31	14	5	16	17	17	12	9	29	2	7	41
b	49	1	0	42	8	0	38	10	2	33	13	4	19	19	12	16	16	18	4	10	36
Σ	402	51	47	368	73	59	348	90	62	279	128	93	175	141	184	130	109	261	90	96	314

TABLE II. ABSOLUTE FREQUENCIES OF JUDGMENTS

Subject IV

C. S.	144			146			148			150			152			154			156		
Series	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l
Ia	41	4	5	36	12	2	28	18	4	26	20	4	20	26	4	10	12	28	8	26	16
b	48	2	0	34	12	4	32	16	2	23	24	3	26	13	11	6	9	35	10	15	25
IIa	50	0	0	47	2	1	39	5	6	36	7	7	13	17	20	8	2	40	2	1	47
b	48	0	2	45	1	4	45	1	4	38	4	8	17	9	24	8	0	42	2	0	48
IIIa	46	4	0	44	5	1	49	1	0	39	3	8	24	11	15	7	1	42	5	1	44
b	48	1	1	46	1	3	49	1	0	37	2	11	18	6	26	4	1	45	5	1	44
IVa	49	1	0	49	1	0	37	8	5	38	7	5	31	11	8	4	0	46	3	3	44
b	49	0	1	47	0	3	41	1	8	27	7	16	30	2	18	0	0	50	1	0	49
Va	49	0	1	47	0	3	47	0	3	4	0	46	11	0	39	1	0	49	8	0	42
b	50	0	0	49	1	0	45	0	5	10	0	40	24	0	26	2	0	48	10	2	38
Σ	478	11	11	444	34	22	412	51	37	278	74	148	214	95	191	50	25	425	54	49	397

Subject V

C. S.	144			146			148			150			152			154			156		
Series	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l
Ia	35	9	6	25	17	8	25	18	7	19	20	11	6	25	19	4	15	31	5	11	34
b	35	13	2	29	16	5	31	12	7	26	15	9	14	16	20	3	9	38	2	14	34
IIa	38	9	3	41	7	2	30	6	14	27	6	17	17	10	23	9	8	33	7	3	40
b	43	7	0	37	10	3	28	8	14	20	8	22	14	14	22	4	13	33	6	5	39
IIIa	41	6	3	28	8	14	29	8	13	29	9	12	5	7	38	6	6	38	5	2	43
b	40	2	8	28	4	18	26	3	21	23	1	26	5	2	43	6	1	43	2	0	48
IVa	28	9	13	27	21	2	12	16	22	10	5	35	0	6	44	1	0	49	1	1	48
b	33	11	6	23	19	8	11	12	27	10	12	28	5	8	37	2	7	41	1	3	46
Va	40	0	10	36	0	14	27	0	23	14	0	36	7	0	43	8	0	42	5	0	45
b	43	2	5	41	0	9	29	4	17	17	3	30	7	0	43	6	1	43	5	1	44
Σ	376	68	56	315	102	83	248	87	165	195	79	226	80	88	332	49	60	391	39	40	421

Subject VI

C. S.	144			146			148			150			152			154			156		
Series	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l
Ia	24	25	1	27	18	5	15	25	10	16	25	9	9	24	17	8	20	22	7	11	32
b	34	14	2	21	27	2	18	29	3	10	27	13	8	24	18	3	13	34	3	19	28
IIa	19	26	5	23	21	6	15	33	2	8	11	31	12	21	17	1	23	26	0	10	40
b	23	17	10	20	21	9	12	21	17	9	12	29	12	10	28	4	5	41	3	4	43
IIIa	18	14	18	17	16	17	7	12	31	6	17	27	28	8	14	29	12	9	15	12	23
b	28	9	13	21	7	22	12	15	23	11	16	23	7	10	33	9	12	29	9	13	28
IVa	26	11	13	34	15	1	11	14	25	8	7	35	16	17	17	9	18	23	1	2	47
b	42	2	6	20	16	14	22	18	10	6	8	36	5	11	34	11	15	24	8	10	32
Va	32	4	14	23	16	11	30	10	10	13	13	24	1	15	34	13	11	26	7	6	37
b	47	2	1	24	15	11	24	13	13	19	15	16	4	15	31	8	13	29	0	1	49
Σ	293	124	83	230	172	98	166	190	144	106	151	243	102	155	243	95	142	263	53	88	359

TABLE III. ABSOLUTE FREQUENCIES OF JUDGMENTS *Subject VII*

C. S.	144			146			148			150			152			154			156		
Series	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l
Ia	39	11	0	34	14	2	14	33	3	19	25	6	11	24	15	3	21	26	2	9	39
b	35	15	0	32	17	1	16	29	5	18	24	8	4	18	28	4	16	30	0	16	34
IIa	40	6	4	43	6	1	29	10	11	26	12	12	9	22	19	5	11	34	3	11	36
b	40	8	2	37	10	3	30	13	7	23	21	6	6	22	22	3	16	31	5	17	28
IIIa	41	5	4	39	7	4	32	7	11	32	6	12	18	20	12	7	13	30	3	16	31
b	41	5	4	36	10	4	25	15	10	26	11	13	19	17	14	9	11	30	4	13	33
IVa	40	9	1	41	7	2	18	21	11	10	19	21	19	21	10	5	16	29	2	10	38
b	35	12	3	40	6	4	21	16	13	15	18	17	14	17	19	8	19	23	5	12	33
Va	42	6	2	40	7	3	34	11	5	24	16	10	8	17	25	6	17	27	3	12	35
b	38	8	4	39	11	0	17	26	7	24	16	10	6	18	26	8	5	37	4	13	33
Σ	391	85	24	381	95	24	236	181	83	217	168	115	114	196	190	58	145	297	31	129	340

Subject VIII

C. S.	144			146			148			150			152			154			156		
Series	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l
Ia	47	2	1	41	7	2	31	6	13	29	3	18	12	1	37	7	1	42	0	1	49
b	40	10	0	31	14	5	18	14	18	31	7	12	7	1	42	1	2	47	0	2	48
IIa	49	0	1	46	0	4	37	0	13	28	0	22	21	0	29	10	0	40	1	0	49
b	50	0	0	50	0	0	42	0	8	27	0	23	12	0	38	7	0	43	2	0	48
IIIa	42	1	7	36	1	13	30	0	18	29	2	19	19	2	29	11	0	39	4	1	45
b	49	0	1	46	0	4	44	0	6	36	1	13	19	0	31	14	0	36	8	0	42
IVa	36	10	4	41	7	2	21	11	18	10	6	34	12	5	33	0	1	49	1	0	49
b	43	5	2	45	2	3	28	3	19	20	3	27	22	1	27	2	0	48	2	0	48
Va	46	0	4	49	0	1	36	0	14	17	0	33	3	0	47	4	0	46	4	0	46
b	48	0	2	47	0	3	40	0	10	19	2	29	3	0	47	5	0	45	0	0	50
Σ	450	28	22	432	31	37	329	34	137	246	24	230	130	10	360	61	4	435	22	4	474

Subject IX

C. S.	144			146			148			150			152			154			156		
Series	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l
Ia	32	12	6	32	9	9	17	21	12	11	20	19	12	12	26	0	14	36	1	5	44
b	33	9	8	31	11	8	20	14	16	15	18	17	5	12	33	2	16	32	2	3	45
IIa	34	10	6	36	5	9	12	17	21	13	13	24	12	7	31	9	7	34	3	3	44
b	35	4	11	28	10	12	20	12	18	14	10	26	8	10	32	6	3	41	2	4	44
IIIa	37	9	4	28	3	19	20	8	22	19	6	25	4	3	43	5	3	42	1	0	49
b	36	11	3	18	18	14	21	7	22	12	11	27	8	4	38	5	1	44	1	6	43
IVa	45	4	1	31	7	12	38	3	9	19	7	24	3	5	42	5	12	33	1	2	47
b	43	4	3	29	5	16	26	8	16	21	7	22	11	6	33	4	6	40	0	2	48
Va	41	2	7	40	4	6	34	3	13	21	3	26	9	6	35	2	4	44	7	1	42
b	39	5	6	30	9	11	37	1	12	19	9	22	7	6	37	4	2	44	1	7	42
Σ	375	70	55	303	81	116	245	94	161	164	104	232	79	71	350	42	68	390	19	33	448

Table IV. Absolute Frequencies of Judgments *Subject X*

C. S.	144			146			148			150			152			154			156		
Series	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l
Ia	38	11	1	40	8	2	32	17	1	20	21	9	10	25	15	6	24	20	7	26	17
b	43	4	3	43	3	4	38	2	10	24	12	14	25	6	19	9	11	30	11	5	34
IIa	49	1	0	50	0	0	39	2	9	35	1	14	23	3	24	10	1	39	8	1	41
b	49	0	1	46	2	2	40	0	10	32	0	18	23	1	26	8	0	42	2	1	47
IIIa	50	0	0	45	0	5	41	1	8	28	1	21	26	2	22	9	0	41	4	0	46
b	50	0	0	45	1	4	47	0	3	35	0	15	23	0	27	10	0	40	2	0	48
IVa	50	0	0	50	0	0	43	0	7	31	1	18	29	0	21	9	1	40	5	1	44
b	48	0	2	49	0	1	38	0	12	33	0	17	21	0	29	5	0	45	10	0	40
Va	49	0	1	48	0	2	43	4	3	40	1	9	15	2	33	14	0	36	3	0	47
b	44	6	0	48	2	0	43	7	0	42	6	2	18	14	18	15	14	21	8	8	34
Σ	470	22	8	464	16	20	404	33	63	320	43	137	213	53	234	95	51	354	60	42	398

Subject XI

C. S.	144			146			148			150			152			154			156		
Series	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l
Ia	43	6	1	42	8	0	36	9	5	15	25	10	9	20	21	5	9	36	1	11	38
b	40	10	0	35	13	2	25	23	2	16	18	16	11	23	16	5	7	38	4	6	40
IIa	43	7	0	46	4	0	27	22	1	22	21	7	9	23	18	3	15	32	1	7	42
b	49	1	0	41	9	0	35	13	2	23	18	9	7	23	20	4	13	33	0	7	43
IIIa	50	0	0	42	8	0	26	23	1	22	24	4	10	18	22	2	12	36	0	6	44
b	36	10	4	39	6	5	26	14	10	18	20	12	12	19	19	11	8	31	5	12	33
IVa	47	2	1	45	5	0	32	17	1	14	23	13	9	30	11	4	22	24	2	7	41
b	48	2	0	46	4	0	28	15	7	17	25	8	7	24	19	3	12	35	2	13	35
Va	46	4	0	41	7	2	36	11	3	25	20	5	10	18	22	6	16	28	0	9	41
b	48	2	0	37	13	0	38	12	0	16	19	15	8	14	28	9	18	23	0	9	41
Σ	450	44	6	414	77	9	309	159	32	188	213	99	92	212	196	52	132	316	15	87	398

Subject XII

C. S.	144			146			148			150			152			154			156		
Series	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l	s	e	l
Ia	48	2	0	43	7	0	43	6	1	6	36	8	2	32	16	1	11	38	0	5	45
b	49	1	0	40	10	0	40	9	1	3	26	21	2	28	20	0	6	44	0	1	49
IIa	47	3	0	40	10	0	20	25	5	7	26	17	2	10	38	0	4	46	0	0	50
b	44	5	1	42	8	0	16	29	5	5	27	18	2	9	39	0	3	47	0	1	49
IIIa	47	3	0	33	15	2	18	23	9	7	23	20	0	10	40	0	2	48	0	0	50
b	49	1	0	31	16	3	14	23	13	4	17	29	0	8	42	0	1	49	0	0	50
IVa	49	1	0	37	13	0	23	23	4	2	32	16	1	19	30	1	7	42	0	1	49
b	48	2	0	25	25	0	20	29	1	8	35	7	1	15	34	0	8	42	0	3	47
Va	47	3	0	33	17	0	16	23	11	7	24	19	0	8	42	0	2	48	0	0	50
b	44	6	0	35	12	3	22	22	6	6	19	25	0	9	41	0	6	44	0	0	50
Σ	472	27	1	359	133	8	232	212	56	55	265	180	10	148	342	2	50	448	0	11	489

TABLE V. RELATIVE FREQUENCIES OF JUDGMENTS.

	I			II			III		
	s	e	l	s	e	l	s	e	l
144	0.878	0.100	0.022	0.860	0.086	0.054	0.804	0.102	0.094
145	0.810	0.218	[—] .029	0.860	0.070	0.071	0.762	0.141	0.097
146	0.774	0.178	0.048	0.688	0.142	0.170	0.736	0.146	0.118
147	0.724	0.162	0.114	0.533	0.200	0.267	0.721	0.155	0.125
148	0.640	0.206	0.154	0.438	0.220	0.342	0.696	0.180	0.124
149	0.523	0.275	0.201	0.380	0.214	0.406	0.643	0.218	0.139
150	0.388	0.320	0.292	0.320	0.204	0.476	0.558	0.256	0.186
151	0.257	0.311	0.433	0.236	0.200	0.564	0.452	0.281	0.268
152	0.152	0.254	0.594	0.138	0.196	0.666	0.350	0.282	0.368
153	0.090	0.187	0.722	0.063	0.175	0.762	0.282	0.258	0.461
154	0.072	0.154	0.774	0.046	0.126	0.828	0.260	0.218	0.522
155	0.074	0.158	0.769	0.079	0.071	0.850	0.259	0.184	0.557
156	0.038	0.100	0.862	0.044	0.108	0.848	0.180	0.192	0.628

	IV			V			VI		
	s	e	l	s	e	l	s	e	l
144	0.956	0.022	0.022	0.752	0.136	0.112	0.586	0.248	0.166
145	0.796	0.029	0.175	0.751	0.184	0.065	0.500	0.255	0.246
146	0.888	0.068	0.044	0.630	0.204	0.166	0.460	0.344	0.196
147	0.915	0.091	[—] .006	0.540	0.195	0.265	0.406	0.392	0.202
148	0.824	0.102	0.074	0.496	0.174	0.330	0.332	0.380	0.288
149	0.679	0.118	0.203	0.459	0.159	0.382	0.260	0.337	0.403
150	0.556	0.148	0.296	0.390	0.158	0.452	0.212	0.302	0.486
151	0.485	0.181	0.335	0.280	0.169	0.552	0.197	0.295	0.508
152	0.428	0.190	0.382	0.160	0.176	0.664	0.204	0.310	0.486
153	0.313	0.147	0.540	0.086	0.163	0.752	0.210	0.317	0.472
154	0.100	0.050	0.850	0.098	0.120	0.782	0.190	0.284	0.526
155	[—] .105	[—] .030	1.136	0.161	0.068	0.770	0.139	0.210	0.651
156	0.108	0.098	0.794	0.078	0.080	0.842	0.106	0.176	0.718

	VII			VII			IX		
	s	e	l	s	e	l	s	e	l
144	0.782	0.170	0.048	0.900	0.056	0.044	0.750	0.140	0.110
145	0.973	0.039	[—] .012	0.966	0.055	[—].020	0.668	0.177	0.155
146	0.762	0.190	0.048	0.864	0.062	0.074	0.606	0.162	0.232
147	0.559	0.322	0.119	0.748	0.068	0.184	0.551	0.165	0.284
148	0.472	0.362	0.166	0.658	0.068	0.274	0.490	0.188	0.322
149	0.457	0.347	0.196	0.581	0.060	0.358	0.415	0.209	0.375
150	0.434	0.336	0.230	0.492	0.048	0.460	0.328	0.208	0.464
151	0.353	0.358	0.290	0.380	0.033	0.587	0.237	0.180	0.583
152	0.228	0.392	0.380	0.260	0.020	0.720	0.158	0.142	0.700
153	0.126	0.384	0.490	0.164	0.011	0.825	0.105	0.122	0.733
154	0.116	0.290	0.594	0.122	0.008	0.870	0.084	0.136	0.780
155	0.176	0.162	0.662	0.119	0.009	0.872	0.079	0.157	0.764
156	0.062	0.258	0.680	0.044	0.008	0.948	0.038	0.066	0.896

	X			XI			XII		
	s	e	l	s	e	l	s	e	l
144	0.940	0.044	0.016	0.900	0.088	0.012	0.944	0.054	0.002
145	0.954	0.019	0.027	0.892	0.085	0.023	0.752	0.244	0.005
146	0.928	0.032	0.040	0.828	0.154	0.018	0.718	0.266	0.016
147	0.876	0.051	0.073	0.732	0.239	0.029	0.632	0.321	0.047
148	0.808	0.066	0.126	0.618	0.318	0.064	0.464	0.424	0.112
149	0.729	0.076	0.194	0.496	0.382	0.122	0.267	0.515	0.218
150	0.640	0.086	0.274	0.376	0.426	0.198	0.110	0.530	0.360
151	0.539	0.097	0.364	0.268	0.443	0.288	0.031	0.446	0.524
152	0.426	0.106	0.468	0.184	0.424	0.392	0.020	0.296	0.684
153	0.305	0.109	0.586	0.130	0.362	0.509	0.029	0.157	0.814
154	0.190	0.102	0.708	0.104	0.264	0.632	0.004	0.100	0.896
155	0.111	0.087	0.802	0.087	0.170	0.743	[—].054	0.119	0.935
156	0.120	0.084	0.796	0.030	0.174	0.796	0.000	0.022	0.978

TABLE VI. ANALYSIS OF EQUALITY JUDGMENTS

	n	Mean	Maximum of function	Position of Maximum; $x_m =$	$0.6745\sqrt{\frac{\Sigma v^2}{6}}$	$0.6745\sqrt{\frac{\Sigma v^2}{n(n-1)}}$
I	656	150.00	0.323	150.33	23.33	0.087
II	541	150.02	0.220	148.28	22.15	0.100
III	688	150.58	0.285	151.60	25.78	0.092
IV	339	150.82	0.192	151.67	16.08	0.117
V	514	149.46	0.204	146.19	22.99	0.110
VI	1022	149.60	0.394	147.30	32.43	0.078
VII	999	150.49	0.396	152.61	31.16	0.077
VIII	135	147.78	0.069	147.48	9.62	0.175
IX	521	149.39	0.212	149.44	22.34	0.105
X	260	151.15	0.111	153.38	15.91	0.150
XI	924	150.63	0.443	150.97	26.31	0.070
XII	846	149.34	0.536	149.65	20.19	0.058

TABLE VII. COEFFICIENTS OF DIVERGENCE

	Shorter	Equal	Longer	Mean		Shorter	Equal	Longer	Mean
I	2.14	3.27	2.09	2.50	VII	1.38	1.41	1.25	1.34
II	1.45	2.37	2.46	2.09	VIII	2.21	1.94	1.82	1.99
III	2.75	1.99	2.53	2.42	IX	1.59	1.73	1.31	1.54
IV	2.19	2.93	2.60	2.57	X	1.68	3.27	1.97	2.31
V	1.75	2.40	2.27	2.14	XI	1.37	1.31	1.38	1.35
VI	2.19	2.02	2.55	2.25	XII	1.35	1.77	1.77	1.62

Table VIII. Interval of Uncertainty

Subject	By Interpolation				By the Method of Just Perceptible Differences							
					Calculated				Observed			
	Lower limit	Upper limit	Inter-val	Middle point	Lower limit	Upper limit	Interval	Middle point	Lower limit	Upper limit	Interval	Middle point
I	149.17	151.41	2.24	150.29	148.57	150.92	2.35	149.75	148.71	151.43	2.72	150.07
II	147.34	150.27	2.93	148.80	147.28	150.06	2.78	148.67	147.41	150.23	2.82	148.82
III	150.35	153.64	3.09	152.09	149.58	147.41	[—]2.17	148.50	150.94	152.49	1.55	151.71
IV	150.78	152.74	1.96	151.76	150.53	151.10	0.57	150.81	150.70	151.93	1.23	151.31
V	147.91	150.48	2.57	149.19	146.97	149.97	3.00	148.47	148.50	150.21	1.71	149.35
VI	144.91	153.51	8.52	149.25	142.96	149.64	6.67	146.30	146.92	151.07	4.15	148.99
VII	147.68	153.09	5.41	150.38	148.08	148.77	0.69	148.42	148.77	152.30	3.53	150.54
VIII	149.91	150.27	0.36	150.09	149.53	150.15	0.62	149.84	149.73	150.24	0.51	149.99
IX	147.83	150.30	2.47	149.06	146.27	149.71	3.44	147.99	147.19	149.95	2.76	148.57
X	151.35	152.27	0.92	151.81	150.67	150.58	[—]0.09	150.62	150.95	152.00	1.05	151.42
XI	148.97	152.93	3.96	150.95	148.91	150.07	1.16	149.49	149.07	152.74	3.67	150.91
XII	147.78	150.85	3.07	149.31	147.50	150.88	3.38	149.19	146.98	150.92	3.94	148.95
Mean	148.67	151.81	3.12	150.24	148.07	149.94	1.87	149.00	148.82	151.29	2.47	150.05